Unicyclic graphs of metric dimension 2

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- We consider only simple, finite, undirected, connected, and nontrivial graphs.
- Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *distance* between two vertices v_1 and v_2, denoted by $d_G(v_1, v_2)$, is the length of the shortest path between them.

Resolving set

- A vertex u of a graph G is said *to resolve* two vertices v_1 and v_2 of G if
$$d_G(u, v_1) \neq d_G(u, v_2).$$
- An ordered vertex set S of G is a *resolving set* if every two distinct vertices of G are resolved by some vertex in S.
- A resolving set is also called a *metric generator*.

Metric dimension

- A *metric basis* of G is a resolving set of minimum cardinality.
- The *metric dimension* of G is the cardinality of a metric basis. We denote the metric dimension of G by $\dim(G)$.

Example 1

Path graph P_n

- A *metric basis* of P_n is a leaf.
- The *metric dimension* of P_n is 1.

Proposition

$\dim(G) = 1$ iff $G = P_n$

Example 2

Proposition

[G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, 2000]
A connected graph G of order $n \geq 2$ has dimension $n - 1$ if and only if $G = K_n$.

Proposition

[G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, 2000]
Let G be a connected graph of order $n \geq 4$. Then $dim(G) = n - 2$ if and only if
$G = K_{s,t}$ $(s,\ t \geq 1)$; $G = K_s + K_t$ $(s \geq 1,\ t \geq 2)$; or
$G = K_s + (K_1 \cup K_t)$ $(s,\ t \geq 1)$.

Metric basis

The concept of a metric basis was first introduced by L. Blumenthal for semimetric spaces. Later, P.J. Slater, and F. Harary and R. Melter extended this idea to graphs by introducing the notions of metric basis and metric dimension for simple, connected graphs.

Applications of metric dimension

Metric dimension as a graph parameter has numerous applications:

- robot navigation;
- combinatorial optimization;
- sonar;
- pharmaceutical chemistry.

In general case the problem to find a metric basis of a graph is NP-hard (M. R. Garey, S. Johnson).

Example 3

Cycle graph C_n

- A *metric basis* of C_n is a set of two neighbours.
- The *metric dimension* of C_n is 2.

Metric dimension of trees

Let T be a tree.

- A **inner vertex** of T is a vertex of degree at least 3.
- An inner vertex v is said to be **close** to a leaf a if there is no other inner vertex w on the unique path between v and a in G; that is, for every other inner vertex w of G, the inequality

$$d_G(a, v) < d_G(a, w)$$

  holds.
- The **terminal degree** of an inner vertex v is the number of leaves for which v is their closest inner vertex.

Theorem

[Slater, 1975, Harary and Melter, 1976]
Let T be a tree that is not a path. Then the metric dimension of T is

$$\dim(T) = \sum_{v \in I(T)} \big(\mathrm{ter}(v) - 1\big),$$

where $I(T)$ is the set of inner vertices of T, and $\mathrm{ter}(v)$ is the terminal degree of v.

Unicyclic graph

- A *unicyclic graph* is a graph that contains exactly one cycle.
- How can we characterize all unicyclic graphs with metric dimension 2?

Let $\hat{G} = (\hat{V}, \hat{E})$ be a subgraph of the unicyclic graph $G = (V, E)$, which is a simple cycle. In other words, $\hat{G}$ is isomorphic to C_m for some positive integer m.

Proposition

[M. Dudenko, B.O.,2017] Let $G = (V, E)$ be a unicyclic graph. If metric dimension of G equals 2 then for any $v \in V \setminus \hat{V}$ the inequality $deg_G(v) \leq 3$ holds.

Proposition

[M. Dudenko, B.O.,2018] Let $G = (V, E)$ be a unicyclic graph. If a metric dimension of G equals 2 then for any $v \in \hat{V}$ the inequality $deg_G(v) \leq 4$ holds.

Projection of a Vertex

A vertex $u \in V \setminus \hat{V}$ of a graph G is said to be *projected* onto a vertex $w \in \hat{V}$ if, for every vertex $q \in \hat{V}$, the inequality

$$d_G(u, w) < d_G(u, q)$$

holds.

Main Vertex

An inner vertex $\hat{G}$, onto which vertices of degree 3 lying outside the cycle are projected, is called a *main vertex*.

Main vertices

Proposition

[M. Dudenko, B.O.,2017] Let $G = (V, E)$ be a unicyclic graph and $dim(G) = 2$. Then there exist at most two main vertices in the graph G.

Gluing

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Fix vertices $v_1 \in V_1$ and $v_2 \in V_2$. A graph G is built from G_1 and G_2 by *gluing* along the vertices v_1 and v_2 if $G = (V, E)$ has the set of vertices $V = V_1 \cup (V_2 \setminus v_2)$ and the set of edges $E = E_1 \cup E_2$ (a vertex v_2 is replaced by v_1 for all edges of G_2).
Roughly speaking, we identify vertices v_1 and v_2 of graphs G_1 and G_2.

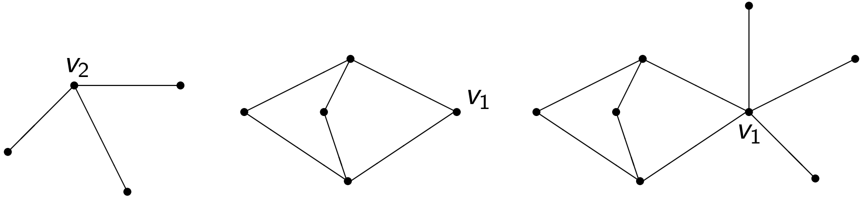

Figure 1: Gluing of two graphs

Inner Vertex Close to a Leaf

- If an inner vertex v is close to two different leaves, then we call v a **two-leaf vertex**.
- If an inner vertex v is close to one leaf, then we call v a **one-leaf vertex**.

Proposition

Let $G = (V, E)$ be a unicyclic graph and $dim(G) = 2$. A vertex $v \in \hat{V}$ with degree 3 is a main vertex of the graph G if an only if v is not a one-leaf vertex.

Even and Odd unicyclic graphs

- A unicyclic graph G is *even*, if $|\hat{V}| = 2k$ for some positive integer k.
- A unicyclic graph G is *odd*, if $|\hat{V}| = 2k + 1$ for some positive integer k.

Basic graph

A unicyclic graph G is said to be a *basic graph* if following conditions hold:

- A. $deg_G(v) \leq 3$, for any vertex v from G ;
- B. for any main vertex v of G there exists exactly one two-leaf vertex projected in v;
- C. G has one or two main vertices of degree 3;
- D. Only main vertices in $\hat{G}$ have degree 3.

Knitting

Let G_1 be a unicyclic graph. Denote by u and w the vertices of degree more than 2 of the subgraph $\hat{G}_1$. A unicyclic graph G is called a *knitting* of the graph G_1 by chains $L_1, \dots, L_r$ if G is obtained from the graph G_1 by gluing vertices of a degree 2 from the cycle of the graph G_1 and leaves of chains $L_1, \dots, L_r$ such that any vertex of degree 2 from the cycle G_1 may be glued with one leaf of one chain L_j, $1 \leq j \leq r$

Theorem (M. Dudenko, B.O. [2017])

An odd unicyclic graph $G = (V, E)$ with two main vertices of degree 3 has metric dimension 2 if and only if one of the following conditions hold:

1. *G is a basic graph;*
2. *G is a knitting of some basic graph G_1 and for any one-leaf vertex w and any adjacent to w vertex a the following inequality holds :*

$$d_G(u, v) + d_G(v, w) + 1 \neq d_G(u, a).$$

An even unicyclic graph $G = (V, E)$ with two main vertices u and v, $|\hat{V}| = 2k$, has metric dimension 2 if and only if one of the conditions 1 or 2 holds and $d(u, v) \neq k$.

Theorem (M. Dudenko, B.O. [2018])

Let $G = (V, E)$ be a unicyclic graph and $dimG = 2$. If the graph G is even, then the degree of any vertex of G less than 4. If the graph G is odd, then the maximum number of vertices of degree 4 is equal to 2.

Unispider graph

An odd unicyclic graph G with $|\hat{V}| = 2k+1$ is said to be a *unispider graph* if the following conditions hold:

1. $deg_G(v) \leq 3$, for any vertex v from $V \setminus \hat{V}$;
2. for any main vertex w of G there exists exactly one two-leaf vertex, that projected in w;
3. in the cycle $\hat{G}$ of the graph G there are exactly two inner vertices w and u;
4. at least one of the main vertices w and u has degree 4, each of vertices w and u is the main vertex or (and) vertex of degree 4 and

$$d_G(w, u) = k.$$

Semiunispider graph

An odd unicyclic graph G with $|\hat{V}| = 2k + 1$ is said to be a *semiunispider graph* if the following conditions hold:

1. for any vertex v from $V \setminus \hat{V}$ $deg_G(v) \leq 3$;
2. in the cycle $\hat{G}$ of the graph G there is exactly one inner vertex w, moreover, $deg(w) = 4$;
3. the vertex w may be main vertex, in this case there exists exactly one two-leaf vertex, that projected in w.

Theorem (M. Dudenko, B.O. [2018])

Let $G = (V, E)$ be a unicyclic graph and $\dim G = 2$. If the graph G is even, then the degree of any vertex of G less than 4. If the graph G is odd, then the maximum number of vertices of degree 4 is equal to 2.

Theorem (M. Dudenko, B.O. [2018])

Let $G = (V, E)$ be a unicyclic graph with vertices of degree 4. Then $dimG = 2$ if and only if one of the following conditions holds:

1. *the graph G is a unispider graph;*
2. *the graph G is a knitting of some unispider graph;*
3. *the graph G is a semiunispider graph;*
4. *the graph G is a knitting of some semiunispider graph;*