On flag-no-square 4-manifolds

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A simplicial complex Δ is **flag** if its faces are exactly the cliques of its 1-skeleton. If additionally, Δ has no induced cycles of length $3 < l < k$ then Δ is k-**large**; where 5-large is also called **flag-no-square**, or **fns** for short.

The flag-no-square condition was introduced and studied in the context of cubical structures on 3-manifolds by L. Siebenmann. Its importance for the geometry of cubical complexes comes from the following observation by Gromov:

A right-angled Coxeter group is word-hyperbolic iff its nerve is a flag-no-square simplicial complex.

It is known that for every simplicial complex of dimension $d \leq 3$ there exists a fns complex homeomorphic to it,

Dranishnikov [Dra99] for $d \leq 2$, and Przytycki and Swiatkowski [PS09] for $d = 3$.

while the 4-sphere and any 4-dimensional homology sphere has no fns triangulation,

Januskiewicz and Swiatkowski [JS03].

Note that every vertex link in a homology manifold (compact, without boundary) is a homology sphere. thus, a d-dimensional (homology) manifold has a fns triangulation if $d < 4$ and has no fns triangulation if $d > 4$.

The following question is open:

Question

Which triangulable 4-manifolds admit a flag-no-square triangulation?

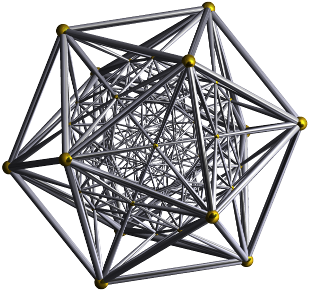
Some lower bounds on the Euler characteristic $\chi(M)$ of 4-manifolds M admitting a fns triangulation are known:

$$\chi(M) \geq f_0(M)$$

if M is a fns triangulation of a 4-manifold.

Kopczýnski, Pak and Przytycki [KPP09].

Hence, e.g. $\chi(M) \geq 18$ by passing to links, as one can show that the icosahedron, which has 12 vertices, minimizes the number of vertices among fns triangulations of the 2-sphere (plus little extra work to reach the 18 bound), and likely $\chi(M) \geq 123$, which would follow if the boundary of the 600-cell minimizes the number of vertices among fns triangulations of the 3-sphere.



On the other hand, very limited constructions of fns 4-manifolds are known, described in [PS09, 4.4(2)]: they are all quotients of the regular simplicial tesselation of the hyperbolic 4-space with all vertex links isomorphic to the boundary complex of the 600–cell, by an appropriate subgroup of its (Coxeter) automorphism group; in particular, the resulting fns manifolds are aspherical.

The star connected sum

Suppose N and M are two disjoint simplicial complexes, and suppose there is a combinatorial isomorphism between the vertex links $\mathrm{lk}_v(N)$ and $\mathrm{lk}_u(M)$ with the isomorphism being induced by some bijection on their vertices $\phi : V(\mathrm{lk}_v(N)) \to V(\mathrm{lk}_u(M))$. We will use the same notation for the combinatorial isomorphism between the vertex links and write it as $\phi : \mathrm{lk}_v(N) \to \mathrm{lk}_u(M)$. Also recall that for a vertex $v \in M$, the antistar $\mathrm{ast}_v(M)$ is the subcomplex consisting of all faces of M not containing v.

Definition

The **star connected sum** $N\#_\phi M$ is the simplicial complex obtained by gluing the antistars $\mathrm{ast}_v(N)$ and $\mathrm{ast}_u(M)$ according to ϕ, namely we take the union of the antistars and identify w with $\phi(w)$ for all $w \in V(\mathrm{lk}_v(N))$.

Lemma 1

Assume N and M are disjoint fns orientable connected 4-manifolds, $v \in V(N)$, $u \in V(M)$ and $\phi : \mathrm{lk}_v(N) \to \mathrm{lk}_u(M)$ a combinatorial isomorphism that reverses orientation. Then

1. *$N\#_\phi M$ is an orientable fns 4-manifold, homeomorphic to the connected sum $N\#M$, and*
2. *$\chi(N\#_\phi M) = \chi(N) + \chi(M) - 2$.*

Lemma 1 implies the following corollary.

Corollary 2

There exist non-aspherical 4-manifolds admitting a fns triangulation.

This settles in the negative Question 5.8(1) in [PS09].

The star handle

Next we use a handle-type construction, similar to the star connected sum: given a connected simplicial manifold N admitting two vertices v and u of graph distance at least 5 with isomorphic vertex links $\mathrm{lk}_v(N)$ and $\mathrm{lk}_u(N)$, denote by $\phi : V(\mathrm{lk}_v(N)) \to V(\mathrm{lk}_u(N))$ a bijection that induces an isomorphism of the links. Under these conditions:

Definition

The **star handled** N, denoted by $h_\phi(N)$, is the simplicial complex[a] obtained from the subcomplex $(N - v) - u$ of N by identifying the links $\mathrm{lk}_v(N)$ and $\mathrm{lk}_u(N)$ according to ϕ, namely we take the quotient of $(N - v) - u$ by identifying w with $\phi(w)$ for all $w \in V(\mathrm{lk}_v(N))$.

[a] The condition on the graph distance $d(u, v) \geq 5$ guarantees that $h_\phi(N)$ is indeed a simplicial complex.

Note that topologically $h_\phi(N)$ is homeomorphic to N with a (hollow) 1-handle attached. The following Lemma follows.

Lemma 3

Let N be a fns orientable connected 4-manifold and $v, u \in V(N)$ of distance at least seven in the graph metric of the 1-skeleton of N, and $\phi : \mathrm{lk}_v(N) \to \mathrm{lk}_u(N)$ a combinatorial isomorphism that reverses orientation. Then

1. *$h_\phi(N)$ is a fns 4-manifold, homeomorphic to N with a hollow 1-handle attached, and*
2. *$\chi(h_\phi(N)) = \chi(N) - 2$.*

All of the fns manifolds constructed in [PS09, 4.4(2)] have even Euler characteristic. The question arises:

Question

Which positive integers can be realized as the Euler characteristic of some fns 4-manifold?

By iterative use of Lemmas 1 and 3, starting with manifolds as above obtained from subgroups with minimal displacement at least 16, we obtain

Proposition 4

For all even s large enough there exists a 4-manifold G_s, admitting a fns triangulation, with $\chi(G_s) = s$.

Given Proposition 4, it is natural to ask

Question

How many combinatorially distinct fns triangulations can a connected 4-manifold admit?

By the result of [KPP09] mentioned above, that $\chi(\Delta) \geq f_0(\Delta)$, indeed there are only finitely many such triangulations.

We give a better estimate: let $t(M)$ be the number of combinatorial types of fns triangulations of a given connected 4-manifold M, and let

$$t(k) = \sum_{\chi(M)=k} t(M)$$

where the summation runs over all connected 4-manifolds of Euler characteristic k.

Theorem 5

1. *For all even k large enough there exists a 4-manifold M_k with $\chi(M_k) = \Theta(k)$ that has at least $k!$ combinatorially distinct fns triangulations. Thus:*

$$t(M_k) \geq 2^{k \log k}.$$

2. *There exists a constant $b > 0$ such that for all k large enough:*

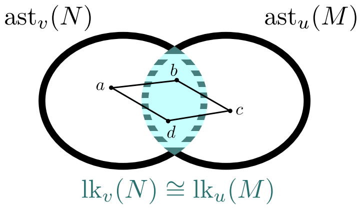
$$t(k) \leq b^{k^{3/2} \log k}.$$

Proof of Lemma 1 I

The star connected sum $N\#_\phi M$ is homeomorphic to the usual connected sum $N\#M$. Note that $\mathrm{ast}_v(N)$, $\mathrm{ast}_u(M)$, $\mathrm{lk}_v(N)$, and $\mathrm{lk}_u(M)$ are flag-no-square since they are induced subcomplexes of the fns simplicial complexes N and M respectively. Note that the union $A \cup B$ of two flag complexes A and B, whose intersection $A \cap B$ is an *induced* subcomplex in both, must be flag as well; thus $N\#_\phi M$ is flag.

Now, assume by contradiction that there is an induced square, namely 4-cycle, $abcd$ in $N\#_\phi M$. As $abcd \not\subseteq \mathrm{ast}_u(M)$ and $abcd \not\subseteq \mathrm{ast}_v(N)$, this induced square has to have exactly one vertex in $\mathrm{ast}_u(M) \setminus \mathrm{ast}_v(N)$ and exactly one vertex in $\mathrm{ast}_v(N) \setminus \mathrm{ast}_u(M)$; as shown in the following figure.

Proof of Lemma 1 II



So if the square has vertices a, b, c, and d, then without loss of generality, we can assume that $a \in \mathrm{ast}_v(N) \setminus \mathrm{lk}_v(N)$, $c \in \mathrm{ast}_u(M) \setminus \mathrm{lk}_u(M)$ and $b, d \in \mathrm{lk}_u(M) = \mathrm{lk}_v(N)$ (under their identification). But then $ubcd$ would be an induced square in M, a contradiction. This proves item (i).
Item (ii) follows directly from (i), by applying the Mayer–Vietoris sequence to our connected sum. Indeed, the Betti numbers satisfy

$$\beta_i(N\#M) = \beta_i(N) + \beta_i(M)$$

for $i = 1, 2, 3$, and $\beta_0(N\#M) = 1 = \beta_4(N\#M)$.

Proof of lemma 3

We know that $h_\phi(N)$ is homeomorphic to N with a hollow 1-handle attached. To see that $h_\phi(N)$ is fns, note that all induced cycles in $h_\phi(N)$ formed by identifying the ends w and $\phi(w)$ in an induced path $w \ldots \phi(w)$ in N have length at least 5, as the corresponding induced path $vw \ldots \phi(w)u$ in N has length at least 7 by assumption. This proves item (i). The Euler characteristic formula $\chi(h_\phi(N)) = \chi(N) - 2$ follows from straightforward computations using the Mayer–Vietoris sequence, proving item (ii).

Proof of Proposition 4

Let M be a connected fns 4-manifold of the construction scheme in [PS09]. All vertex links of M are isomorphic to the boundary complex of the 600–cell, and the diameter of the graph of M is at least 8.
Take two vertices u, v which are diametrically apart, and let $P = uabu'c \dots dv'ev$ be an induced path realizing the diameter of G.
Let $h(M)$ be the star handled M (denote $\chi(M) = m$).
Take a "row" of k copies of $h(M)$ with

$$k(m-4)+2 \leq s < (k+1)(m-4)+2,$$

If the equality $k(m-4)+2 = s$ above holds then $\chi(M') = s$, and we are done. Otherwise, replace (the last) $\frac{s-k(m-4)-2}{2}$ copies of $h(M)$ in the row with copies of M.
To summarize, we constructed connected fns 4-manifolds G_s with $\chi(G_s) = s$ for all even $s \geq \frac{(m-6)(m-4)}{2} + 2$.

Proof of Theorem 5(ii)

Recall that a fns 4-manifolds with Euler characteristic k has at most k vertices by [KPP09], and notice that a fns 4-manifold is determined by its 1-skeleton G (by flagness), and (i) G has no induced C_4 and (ii) G has largest clique size 5. From [GHS02, Theorem 1] it follows that a graph with n vertices satisfying (i) and (ii) has at most $\frac{5}{\sqrt{2}}n^{1.5}$ edges.

Let

$$F = \left\{ G \;\middle|\; |V(G)| \leq k, |E(G)| \leq \frac{5}{\sqrt{2}}|V(G)|^{1.5} \right\}.$$

Then

$$t(k) \leq |F| = \sum_{y \leq k} \sum_{i \leq \frac{5}{\sqrt{2}}y^{1.5}} \binom{\binom{y}{2}}{i} < k \sum_{i \leq \frac{5}{\sqrt{2}}k^{1.5}} \binom{\binom{k}{2}}{i} \cdots < e^{9k^{1.5} \cdot \ln k}$$

and hence we obtain the claimed bound with constant $b = e^9$.

Proof of Theorem 5(i): 4-manifolds with factorially many different fns triangulations

The idea behind our construction is to realize geometrically the cycle structure of a permutation.

We construct certain decorated building blocks to represent $1, \ldots, k$, glue them in a row via the star connected sum operation, and then we further glue a decoration of the block corresponding to i to a decoration of the block corresponding to j whenever $\sigma(i) = j$, for $\sigma \in S_k$.

We then show that the triangulations T_σ and T_π thus obtained are combinatorially distinct whenever $\sigma \neq \pi$, for permutations $\sigma, \pi \in S_k$.

Building blocks

Let M be our starting connected 4-manifold, denote $\chi(M) = n$, and let T_M be its fns triangulation, and $G := G(T_M)$ the graph (1-skeleton) of T_M. Consider two vertices $u, v \in V(G)$ diametrically apart, and denote $d(u, v) = \text{diam}(G) =: d$.

We can have d as large as we want, by letting $M = \mathbb{H}^4/K$ for an appropriate subgroup K as explained in the Introduction, and in the resulted triangulation T_M all vertex links are isomorphic to the boundary of the 600-cell. How large we need d to be will be determined along the proof.

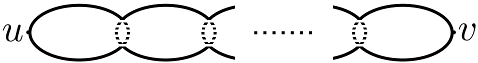
Let w be a third vertex (approximately) half way between u and v. Explicitly: take w on a path of length d from u to v so that $d(u, w) = \lceil \frac{d}{2} \rceil$ and $d(v, w) = \lfloor \frac{d}{2} \rfloor$.

Building blocks

The **row** R_a of length a is the fns 4-manifold

$$R_a := \underbrace{T_M \# T_M \# \ldots \# T_M}_{a \text{ times}}$$

A **crucial** choice here is to perform each of the gluings with the identity map induced on the link of the vertices where the gluing occurs



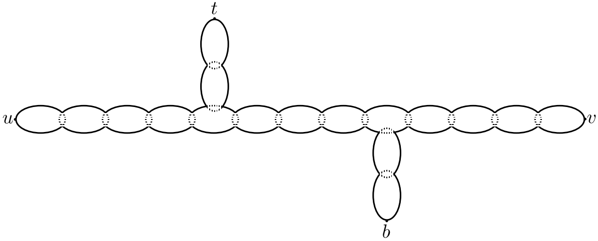
Our crucial choice of gluing allows us to have a good estimate on the diameter of a row R_a:

$$2(d-1)+(a-2)(d-2) = d(u,v) \leq \mathrm{diam}(R_a) \leq 2(d+3)+(a-2)(d-2)$$

Thus, we know the diameter of R_a up to constant error, of 8.

Building blocks

A second building block employs the third distinguished vertex w: The 4-manifold E is obtained by applying the star connected sum construction to 17 copies of M (of T_M to be precise) according to the following scheme:



$$\mathrm{diam}(E) \leq 2(d+3) + 2(d+1) + 9(d-2) = 13d - 10$$

Building blocks

Let $E_1, \ldots, E_k$ be k copies of E, and denote the corresponding four distinguished vertices of E_i by u_i, v_i, t_i, b_i. We define a "row" of star connected sums:

$$N_k := R_{15k} \# E_1 \# \ldots \# E_k \# R_{14k},$$

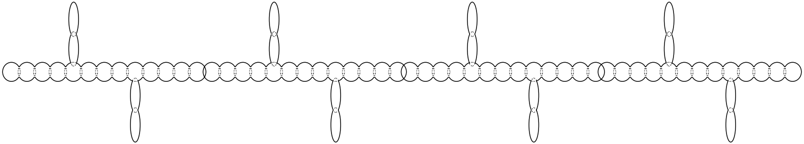
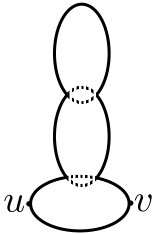


Figure: N_4

Constructing T_σ

Given a permutation $\sigma \in S_k$, we attach k 1-handles to N_k and obtain a manifold M_k with a fns triangulation T_σ. Explicitly, for each $1 \leq i \leq k$ we glue a copy of H (as in the figure below) along its u to t_i of N_k, and along its v to b_j of N_k, where $\sigma(i) = j$.



By Lemmas 1 and 3, all the fns complexes T_σ thus obtained, for $\sigma \in S_k$, triangulate the same topological manifold, denoted by M_k.

Constructing T_σ

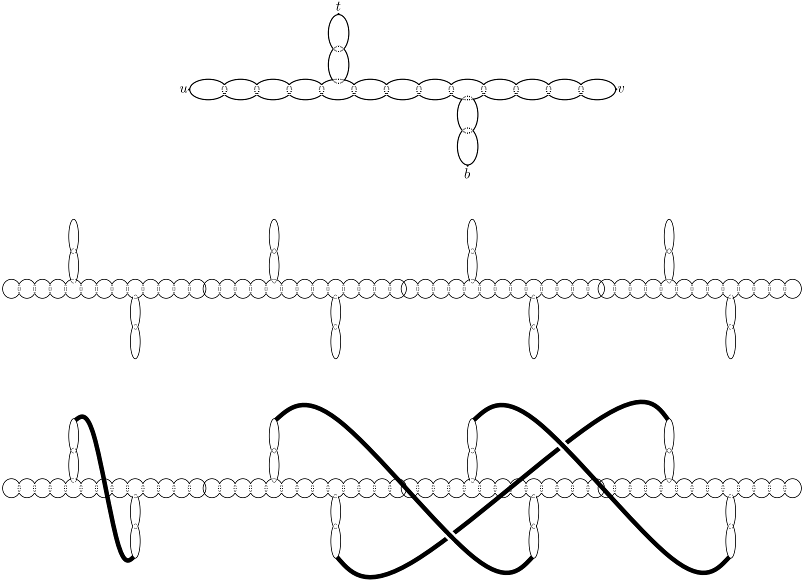
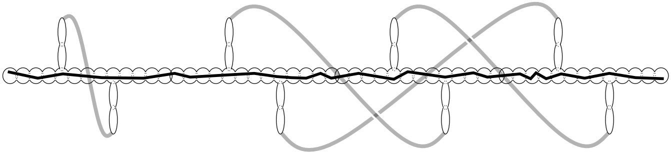


Figure: T_σ for $\sigma = (2, 3, 4) \in S_4$

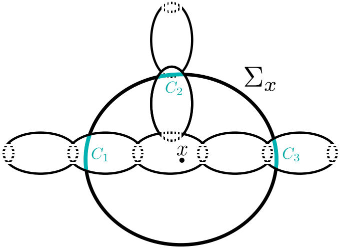
Reconstructing σ from T_σ

Let $G = G(T)$ be the graph of T.

1. Find two vertices $s, e \in V(G)$ diametrically apart in G.
2. Find a path from s to e through the "spine".



3. Walk along the path with a "lens" to identify junctions.



4. Short distance between the junctions u_i and v_j means $\sigma(i) = j$.

Reconstructing σ from T_σ: determination of junctions I

A key ingredient for the reconstruction is the determination of the junctions. A copy of M is called a **junction** if it has been glued on all of its three distinguished vertices u, v and w. A junction in a copy of H is called a **handle-junction**. To identify the junctions of T, consider for every vertex $x \in V(G)$ the **sphere of radius** $\lceil \frac{3}{2}d \rceil$ **centered at** x:

$$\Sigma_x = \left\{ y \in V(G) \,\middle|\, d(y,x) = \left\lceil \frac{3}{2}d \right\rceil \right\}.$$

Let $B_l(x)$ denote the set of vertices in G of graph distance at most l from x. Note that Σ_x is the boundary of $B_{\lceil \frac{3}{2}d \rceil}(x)$.

Reconstructing σ from T_σ: determination of junctions II

Lemma 6

Let $d \geq 77$.

1. *If x is in a junction then Σ_x is partitioned into three sets, called* **clusters***:*

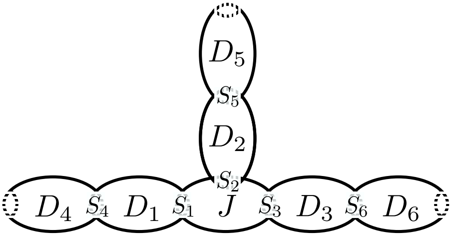
$$\Sigma_x = C_1 \cup C_2 \cup C_3$$

   where two vertices from the same cluster have distance at most $d + 18$, and two vertices from different clusters have distance at least $\frac{3}{2}d - 20$ (which is larger then $d + 18$ by our assumption on d).

2. *Assume x is in a junction J. Then J is a handle-junction if and only if $G \setminus B_{\lceil \frac{3}{2}d \rceil}(x)$ is disconnected with one small component (of at most $2|V(T_M)|$ vertices), and a second component containing all other vertices.*

Reconstructing σ from T_σ: determination of junctions

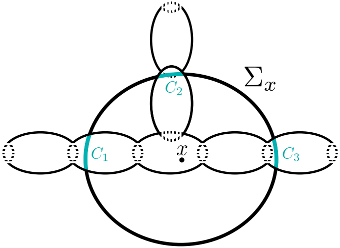
proof: Let us consider the $2d - 4$ neighborhood N of a junction J:



Since $x \in J$ is at distance at most $d + 9$ from any vertex of $S_1 \cup S_2 \cup S_3$ (recall this distance estimate comes from the fact that the diameter of the 600-cell graph is 5), each vertex in Σ_x must lie outside J, at least $\frac{3}{2}d - (d + 9) = \frac{1}{2}d - 9$ away from J, for $d \geq 19$. On the other hand, any vertex of N not in J must belong to $D_1 \cup \cdots \cup D_6$.

Reconstructing σ from T_σ: determination of junctions I

Hence the sphere Σ_x consists of the three light blue arcs in the figure below:



which satisfy the conditions in item (1) for clusters:

$$C_1 = \Sigma_x \cap (D_1 \cup D_4)$$
$$C_2 = \Sigma_x \cap (D_2 \cup D_5)$$
$$C_3 = \Sigma_x \cap (D_3 \cup D_6)$$

Reconstructing σ from T_σ: determination of junctions II

Indeed, consider two vertices p, q in C_1. If both belong to D_1 or both belong to D_4, then they are at most $d + 6$ away from each other.

Suppose $p \in D_1$ and $q \in D_4$. We have

- $\lceil \frac{3}{2} d \rceil = d(p, x) \leq d(p, S_1) + \mathrm{diam}(S_1) + d(S_1, x)$
- $\lceil \frac{3}{2} d \rceil = d(q, x) \geq d(x, S_1) + d(S_1, S_4) + d(S_4, q)$
- $d(p, q) \leq d(p, S_4) + \mathrm{diam}(S_4) + d(S_4, q)$

Substituting the estimates that we have into the first inequality we get $\lceil \frac{d}{2} \rceil - 8 \leq d(x, S_1)$. Then plugging the obtained inequality into the second inequality we get $d(S_4, q) \leq 10$, and finally, the third inequality yields $d(p, q) \leq d + 18$ as promised.

The same argument works for C_2 and C_3.

Reconstructing σ from T_σ: determination of junctions III

Next, consider two vertices from different clusters, say $u_1 \in C_1$ and $u_2 \in C_2$. Since a shortest path from u_1 to u_2 must pass through S_1 and S_2 we have:

$$\begin{aligned} d(u_1, u_2) &\geq d(u_1, S_1) + d(S_1, S_2) + d(S_2, u_2) \\ &\geq \left(\frac{1}{2}d - 9\right) + \left(\frac{1}{2}d - 2\right) + \left(\frac{1}{2}d - 9\right) = \frac{3}{2}d - 20. \end{aligned}$$

Similarly, the same estimate holds for the other two choices of a pair of clusters C_i. This completes the proof of (1).
To prove (2) notice that if x is in a handle-junction J then $G \setminus B_{\lceil \frac{3}{2}d \rceil}(x)$ is indeed composed of two connected components: the top part of an H, and the rest of the graph G. While, if J is a non-handle-junction then either $G \setminus B_{\lceil \frac{3}{2}d \rceil}(x)$ is connected, or $G \setminus B_{\lceil \frac{3}{2}d \rceil}(x)$ has two "large" components: one contains the left R_{15k}, while the other contains the right R_{14k}; each has more than $2|V(T_M)|$ vertices. $\square$

Remark 7

The conclusion in item (1) of Lemma 6 could hold even for vertices not in a junction but only close to a junction (for instance, vertices in D_1, close to S_1). Call these **junction vertices**. We can still identify a vertex that is actually **in** a junction by considering a maximal sequence of consecutive junction vertices along an induced path in G passing from left to right (i.e., through the sequence of components D_4, D_1, J, D_3, D_6) and taking the middle vertex in this sequence.

Note that for every path P from s to e that intersects each E_i only in its row part, each vertex $x \in P$ satisfies that $G \setminus B_{\lceil \frac{3}{2} d \rceil}(x)$ is either connected or has two large connected components, each of size larger than $2|V(T_M)|$. Furthermore, every shortest path from s to e among those **not** intersecting any copy of H is of the same form as P above.

Reconstructing σ from T_σ: the spine and the junction vertices

Thus, using Lemma 6, we may identify a shortest path π from s to e among those that do not pass through handle-junctions:

$$\pi = (s = p_1, p_2, \ldots, p_r = e) .$$

Walking along π, and considering the sequences of spheres $(\Sigma_{p_i})_{i=1}^{r}$ we will encounter a subsequence of consecutive junction vertices (maximal w.r.t. inclusion) after about $15k(d-2)$ steps. Its middle vertex u_1 is in a junction $J_{1,1}$ of E_1. Continuing the sequence of spheres we will have a short junction-free subsequence, followed by a second (maximal) subsequence of consecutive junction vertices – its middle vertex v_1 is in a junction $J_{2,1}$ of E_1. Proceeding in this manner, we distinguish the vertices

$$u_1, v_1, u_2, v_2, \ldots, u_k, v_k$$

where u_i lies in junction $J_{1,i}$ of E_i, and v_i lies in junction $J_{2,i}$ of E_i, for each $1 \le i \le k$.

Reconstructing σ from T_σ: the end I

We can now recover σ:

Lemma 8

Let $d \geq 77$. For each $1 \leq i, j \leq k$ we have:

1. *If $\sigma(i) = j$, then $d(u_i, v_j) \leq 7d + 51$.*
2. *If $\sigma(i) \neq j \neq i$, then $d(u_i, v_j) > 8d - 16$.*

Thus, for all $1 \leq i \leq k$, if there exists $j \neq i$ such that $d(u_i, v_j) \leq 7d + 51$ then this j is unique and $\sigma(i) = j$, else $\sigma(i) = i$.

As discussed above, for $d \geq 77$ the junction vertices $u_i \in J_{1,i}$ and $v_j \in J_{2,j}$ are well defined for $1 \leq i, j \leq k$. If $\sigma(i) = j$, then consider a shortest path from u_i to v_j among those that pass through the handle H connecting the copy of M containing t_i to the copy of M containing b_j. Such a path starts with u_i, crosses the junction $J_{1,i}$ of E_i along at most $d + 9$ edges, then crosses the next two copies of M reaching the handle H along at most $2(d + 6)$ edges, then crosses H along at most $d + 9$ edges, then crosses the next two

Reconstructing σ from T_σ: the end II

copies of M reaching the junction $J_{2,j}$ of E_j along at most $2(d+6)$ edges, and continues in this $J_{2,j}$ to v_j along at most $d+9$ edges. Altogether this path has length at most $7d+51$, proving item (1). Next, assume $\sigma(i) \neq j \neq i$. Then, a shortest path from u_i to v_j either crosses at least eight consecutive non-junction copies of M in a row – passing from some E_k to $E_{k\pm1}$ – thus such path has length at least $8(d-2)$, or it passes through at least two handles, yielding a path of length at least $13(d-2)$. This proves item (2). As $d \geq 77$, then $8(d-2) > 7d+51$, and the recovery of the unique permutation σ for which $T = T_\sigma$ is complete.

End of the proof

To finish the proof of Theorem 5(i) let us observe that the Euler characteristic of M_k is indeed linear in k: denote $n = \chi(M)$. First, notice that attaching a handle H contributes $\chi(H) - 4 = 3n - 8$ to χ, by Lemmas 1(ii) and 3(ii). Thus:

$$\chi(M_k) = \chi(N_k) + k(3n - 8)$$

Now, we compute $\chi(N_k)$ using Lemmas 1(ii) and 3(ii):

$$\begin{aligned}\chi(N_k) &= \chi(R_{15k} \# E_1 \# \ldots \# E_k \# R_{14k}) \\ &= \chi(R_{15k}) + k\chi(E) + \chi(R_{14k}) - 2(k+1) \\ &= (46n - 92)k + 2\end{aligned}$$

so we conclude

$$\chi(M_k) = (46n - 92)k + 2 + k(3n - 8) = (49n - 100)k + 2 = \Theta(k)$$

as claimed.

Concluding remarks I

In view of Corollary 4, we ask:

Problem

Is there a 4-manifold of odd Euler characteristic that admits a fns triangulation?

If the answer is Yes, with a construction admitting a vertex whose link is isomorphic to the boundary of the 600-cell, then gluing it to the fns manifolds G_s of Corollary 4 via a star connected sum along the link of such vertex, would yield connected fns 4-manifolds realizing every large enough integer as their Euler characteristic.

Concluding remarks II

In view of Theorem 5, we ask:

Problem

Is the number of fns triangulations of a 4-manifold M_i super-factorial in $\chi(M_i)$ for a suitable sequence of manifolds $(M_i)_i$?

Similarly, can the upper bound of Theorem 5 on $t(M_i)$ be improved? Is $t(x)$ of larger order of magnitute than any $t(M)$ where $\chi(M) = x$ and x tends to infinity?

Next, we consider the piecewise linear structure of our constructed manifolds:

Problem

Are the different combinatorial triangulations T_σ of the 4-manifold M_k we get in Theorem 5 PL-homeomorphic?

Thanks for your attention!

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