Coherent configurations
Lecture 2
Coherent configurations and coherent algebras

Mikhail Klin (Ben-Gurion University)

September 2025

Coherent algebras

- The essential properties of centralizer algebras are axiomatized.
- We get definition of coherent algebras.
- Relational language of coherent configurations.
- Origins:
  - B. Weisfeiler & A. Leman (Moscow, 1968): cellular algebras;
  - D. Higman (USA, 1970): coherent configurations.

Combinatorial imitation

- Coherent configurations (algebras) provide a basis for combinatorial imitation of permutation groups.
- Structure constants aka intersection numbers have a definite combinatorial spirit: numbers of walks in color graph.
- How far does this imitation go?

Axiomatization

- Let $\mathcal{W} \subseteq M_n(\mathbb{C})$ be a matrix algebra over $\mathbb{C}$ which fulfills the following requirements:

CA1. Considered as a vector space over $\mathbb{C}$, the algebra $\mathcal{W}$ has a basis $\{A_1, A_2, \ldots, A_r\}$ where each A_i is a $(0,1)$-matrix, $1 \leq i \leq r$;

CA2. $\sum_{i=1}^{r} A_i = J_n$, where J_n is the matrix of order n every entry of which is equal to 1;

CA3. For each $i \in \{1, 2, \ldots, r\}$ there is an $i' \in \{1, 2, \ldots, r\}$ such that $A_i^t = A_{i'}$;

CA4. The identity matrix I_n belongs to $\mathcal{W}$.

Standard basis

- In this case we call $\mathcal{W}$ a *coherent algebra with standard basis* $\{A_1, A_2, \ldots, A_r\}$.
- We indicate this all at once by writing

$$\mathcal{W} = \langle A_1, A_2, \ldots, A_r \rangle.$$

Coherent rings

- Given a coherent algebra $\mathcal{W}$, the set $\mathcal{W}' = \mathcal{W} \cap M_n(\mathbb{Z})$ of all integer-valued matrices from $\mathcal{W}$ is both a ring and a $\mathbb{Z}$-module.
- A matrix ring with this property is called a *coherent ring*, denoted by $\mathcal{W}' = \langle A_1, \ldots, A_r \rangle_{\mathbb{Z}}$.

Cellular algebras

- If we suppress axiom CA4 in the definition of coherent algebra, we obtain what is called a *cellular algebra*. The term “cellular algebra” can be traced to the Soviet school of algebraic combinatorics, having been introduced by B.Ju. Weisfeiler and A.A. Leman in 1968.
- Thus, a cellular algebra is a matrix algebra over $\mathbb{C}$ which satisfies axioms CA1–CA3.

Relational language

- Now we shall formulate coherent algebras in terms of a relational language.
- Let $X = \{1, 2, \ldots, n\}$, and let us consider a collection $\mathcal{R} = \{R_1, R_2, \ldots, R_r\}$ of binary relations on X.

Axiomatization

- If the following conditions hold:
  - CC1. $R_i \cap R_j = \emptyset$ for $1 \leq i \neq j \leq r$;
  - CC2. $\cup_{i-1}^{r} R_i = X^2$;
  - CC3. For each $i \in \{1, 2, \ldots, r\}$ there is an $i' \in \{1, 2, \ldots, r\}$ such that $R_i^t = R_{i'}$;
  - CC4. There exists a subset $I' \subseteq \{1, \ldots r\}$ such that $\cup_{i \in I'} R_i = \Delta$, (here $\Delta = \{(x, x) \mid x \in X\}$);
  - CC5. For each $i, j, k \in \{1, 2, \ldots, r\}$ the number of elements $z \in X$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant provided that $(x, y) \in R_k$. We denote this constant by p_{ij}^k.
- $\mathfrak{M} = (X, \mathcal{R})$ is a coherent configuration.

Equivalence of axiomatizations

- Given a coherent configuration $\mathfrak{M} = (X, \mathcal{R})$, we consider the graph $\Gamma_i = (X, R_i)$ defined by the relation R_i and we let $A_i = A(\Gamma_i)$ be its adjacency matrix.
- In this case $\mathcal{W} = \langle A_1, A_2, \ldots, A_r \rangle$ is indeed a coherent algebra.
- Note a natural correspondence between axioms CA1-CA4 of a coherent algebra and axioms CC1-CC4 given above.

Fifth axiom

- Axiom CC5 ensures that each product A_iA_j is a linear combination of the matrices $A_1, \ldots, A_r$. (More precisely, axiom CC5 is equivalent to the fact that a coherent algebra is, by definition, a matrix algebra.)

Equivalence of axiomatizations

- Conversely, given a coherent algebra $\mathcal{W}$ we can easily construct a corresponding coherent configuration $\mathfrak{M}$ by interpreting each matrix A_i as the adjacency matrix of a graph having arc set R_i.

2-orbits

- An important class of examples of coherent configurations is provided by the so-called 2-orbits of permutation groups.
- This terminology was introduced by Wielandt and is equivalent to the language of centralizer algebras of permutation groups.

2-orbits

- Let (G, Ω) be a permutation group. We consider the naturally induced action of G on Ω^2 as follows: For $(a, b) \in \Omega^2$ and $g \in G$ we define $(a, b)^g = (a^g, b^g)$.
- The set of orbits of (G, Ω^2) will be denoted by 2-orb(G, Ω) (following Wielandt).
- We shall refer to the elements of 2-orb(G, Ω) as 2-orbits of (G, Ω).

2-orbits

- Given any permutation group (G, Ω), the pair $(\Omega, \text{2-orb}(G, \Omega))$ is clearly a coherent configuration.
- One may establish this fact by checking axioms CC1–CC5 directly, although this is not necessary.
- Indeed, one need only observe that $(\Omega, \text{2-orb}(G, \Omega))$ corresponds to the coherent algebra $V(G, \Omega)$.

Example 2.1 (K_4)

Consider the complete graph K_4 with vertex set $X_1 = \{1, 2, 3, 4\}$. We label the edges of K_4 by the elements of $X_2 = \{5, 6, 7, 8, 9, 10\}$ in a fixed but arbitrary manner, and we designate by $e(x)$ the edge of K_4 which carries the label $x \in X_2$. Then the symmetric group $S_4 = S(X_1)$ acts intransitively on the set $X := X_1 \cup X_2$ with orbits X_1 and X_2.

Example 2.1 (cont.)

- The 2-orbits of this action:
- $R_1 = \{(x,x) \mid x \in X_1\}$, $R_2 = \{(x,x) \mid x \in X_2\}$,
  $R_3 = \{(x,y) \mid x,y \in X_1,\ x \neq y\}$,
  $R_4 = \{(x,y) \mid x,y \in X_2,\ x \neq y,\ e(x) \cap e(y) \neq \emptyset\}$,
  $R_5 = \{(x,y) \mid x,y \in X_2,\ e(x) \cap e(y) = \emptyset\}$,
  $R_6 = \{(x,y) \mid x \in X_1,\ y \in X_2,\ x \in e(y)\}$,
  $R_7 = \{(x,y) \mid x \in X_1,\ y \in X_2,\ x \notin e(y)\}$,
  $R_8 = \{(x,y) \mid x \in X_2,\ y \in X_1,\ y \in e(x)\}$,
  $R_9 = \{(x,y) \mid x \in X_2,\ y \in X_1,\ y \notin e(x)\}$.
- Resulting from this action, one obtains the coherent configuration $\mathfrak{M} = (X, \mathcal{R})$, where $\mathcal{R} = \{R_1, R_2, \ldots, R_9\}$.

SH-product

- A non-standard matrix multiplication called *Schur-Hadamard multiplication* (*SH-multiplication*, for short).
- Let $A = (a_{ij})$ and $B = (b_{ij})$ be two square matrices of order n, and define

$$c_{ij} = a_{ij} b_{ij}, \quad 1 \leq i, j \leq n.$$

- The matrix $C = (c_{ij})$ is called the *Schur-Hadamard product* of A and B and it is denoted by $C = A \circ B$.

Coherent algebras

- Every coherent algebra is closed with respect to SH-multiplication.
- By linearity one need only verify this for Schur-Hadamard products of basis matrices.
- As these are $(0, 1)$-matrices, one clearly has $A_i \circ A_i = A_i$, and $A_i \circ A_j = O$ for all $i \neq j$, where O denotes the matrix of order n all of whose entries are equal to 0.

Schur-Wielandt Principle

- Let $\mathcal{W}$ be a coherent algebra, and let $X = (x_{ij}) \in \mathcal{W}$.
- For arbitrary $\nu \in \mathbb{C}$ define $Y_\nu(X) = (y_{ij})$ $\big($cross-section of X by $\nu\big)$ by

$$y_{ij} = \begin{cases} \nu & \text{if} \quad x_{ij} = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

- Then $Y_\nu(X) \in \mathcal{W}$ for all $\nu \in \mathbb{C}$.

Proof

- If $x_{ij} \neq \nu$ for all i, j then $Y_\nu(X) = O \in \mathcal{W}$.
- So assume $x_{st} = \nu$ and define the matrix $T_1 = X - X \circ A_{k_1}$, where A_{k_1} is the unique basis matrix for which $(A_{k_1})_{st} = 1$.
- Clearly T_1 is in $\mathcal{W}$ and has fewer entries equal to ν than did X.

Proof

- One repeats the procedure by defining $T_2 = T_1 - T_1 \circ A_{k_2} (= X - (X \circ A_{k_1} + X \circ A_{k_2}))$ and so on, until the matrix T_q obtained on the qth iteration is free from entries equal to ν.
- Then one has

$$Y_\nu(X) = X \circ A_{k_1} + X \circ A_{k_2} + \cdots + X \circ A_{k_q}$$
$$(= \nu(A_{k1} + A_{k_2} + \cdots + A_{k_q})),$$

which is clearly an element of $\mathcal{W}$.

Corollary

- Assume $\nu \neq 0$.
- For any matrix $X \in \mathcal{W}$, there exists a subset $K = \{k_1, k_2, \ldots, k_q\}$ of $\{1, 2, \ldots, r\}$ for which

$$\frac{1}{\nu} Y_\nu(X) = \sum_{k_i \in K} A_{k_i}.$$

Proposition 2.1

A subspace $\mathcal{W}$ of $M_n(\mathbb{C})$ is a coherent algebra if and only if $\mathcal{W}$ contains the matrices I_n and J_n and is closed with respect to the operations of matrix multiplication, SH-multiplication, and conjugate-transposition.

- Proof will be discussed in the exercise meeting.

Proposition 2.2

Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two coherent algebras of order n. Then their intersection $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ is also a coherent algebra of order n.

- Proof: this is an immediate corollary of Proposition 2.1.

Coherent subalgebras

- Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be coherent algebras with $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Then $\mathcal{W}_1$ is called a *coherent subalgebra* of $\mathcal{W}_2$.
- Let $A \in M_n(\mathbb{C})$ be arbitrary. The minimal coherent algebra which contains A is called the *coherent algebra generated by* A and will be denoted by $\langle\langle A \rangle\rangle$.

Coherent subalgebras

- In similar fashion, one more generally defines the coherent algebra $\langle\langle A_1, A_2, \ldots, A_k \rangle\rangle$ generated by the matrices $A_1, A_2, \ldots, A_k$.
- Clearly, if $\{A_1, A_2, \ldots, A_r\}$ is the standard basis of a coherent algebra $\mathcal{W}$, then one has $\langle\langle A_1, A_2, \ldots, A_r \rangle\rangle = \langle A_1, A_2, \ldots, A_r \rangle = \mathcal{W}$.

Coherent subalgebras

- The definitions given above make sense since, by Proposition 2.2, any intersection of coherent algebras is again a coherent algebra.
- $\langle\langle A \rangle\rangle$ may be interpreted as the intersection of all coherent subalgebras of $M_n(\mathbb{C})$ which contain A.

Coherent subalgebras

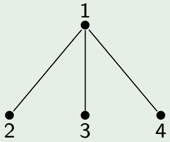
- Let $\mathcal{C}(A)$ denote the set of all such subalgebras.
- $\mathcal{C}(A) \neq \emptyset$ since it contains the coherent algebra $\mathcal{V}_{\mathbb{C}}(\{e\}, \{1, 2, \ldots, n\}) = M_n(\mathbb{C})$.
- Thus

$$\langle\langle A \rangle\rangle = \bigcap_{\mathcal{W} \in \mathcal{C}(A)} \mathcal{W}.$$

Computation

- At the moment we do not wish to discuss in evident form, various algorithms for constructing $\langle\langle A\rangle\rangle$.
- Nonetheless, we mention an extremely effective algorithm which accomplishes this, namely *Weisfeiler-Leman stabilization*.
- The following examples demonstrate how $\langle\langle A\rangle\rangle$ can be constructed using certain tricks based mainly on the Schur-Wielandt principle.

Example 2.2



- Let Γ be the above graph and let $A = A(\Gamma)$ be its adjacency matrix.
- We wish to determine $\mathcal{W} = \langle\langle A \rangle\rangle$.
- Clearly, $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Example 2.2 (cont.)

- All we know from the outset is that $A, I_4, J_4 \in \langle\langle A \rangle\rangle$.
- Since $\langle\langle A \rangle\rangle$ is a linear space, we get $\bar{A} \in \langle\langle A \rangle\rangle$ where

$$\bar{A} = J_4 - I_4 - A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Example 2.2 (cont.)

- Now, since $\langle\langle A\rangle\rangle$ is closed under matrix multiplication, it must contain the two additional matrices:
- $A^2 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ and
- $\bar{A}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$.

Example 2.2 (cont.)

- Applying the Schur-Wielandt principle, we further obtain $B_1, B_2 \in \langle\langle A \rangle\rangle$, where
- $B_1 = \frac{1}{3} Y_3(A^2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and
- $B_2 = \frac{1}{2} Y_2(\bar{A}^2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Example 2.2 (cont.)

- Redefining $B_3 = A$ and $B_4 = \bar{A}$, we now have four $(0,1)$-matrices B_1, B_2, B_3, B_4 of $\langle\langle A \rangle\rangle$ which have mutually disjoint support and sum to J_4.
- However, these four matrices do not constitute a basis for $\langle\langle A \rangle\rangle$, since $\langle\langle A \rangle\rangle$ must additionally contain all products of the form $B_i B_j$ for $i, j \in \{1, 2, 3, 4\}$.

Example 2.2 (cont.)

- In particular, $B_1 B_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B_3 B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ must be elements of $\langle\langle A \rangle\rangle$, resulting in a "desymmetrization" of the matrix A.

Example 2.2 (cont.)

- This gives a new set of five matrices:
  $C_1 = B_1,\ C_2 = B_2,\ C_3 = B_1A,\ C_4 = AB_1,$
  $C_5 = B_4$, which turns out to provide the desired basis for $\langle\langle A\rangle\rangle$.
- Indeed, all matrix products of the form C_iC_j are elements of $\langle C_1, \ldots, C_5\rangle$, as is readily verifiable from the following table of products.

Example 2.2 (cont.)

	C_1	C_2	C_3	C_4	C_5
C_1	C_1	O	C_3	O	O
C_2	O	C_2	O	C_4	C_5
C_3	O	C_3	O	$3\,C_1$	$2\,C_3$
C_4	C_4	O	$C_2 + C_5$	O	O
C_5	O	C_5	O	$2\,C_4$	$2\,C_2 + C_5$

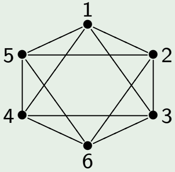
Example 2.2 (cont.)

- To this point we have shown that $\langle C_1, \ldots, C_5 \rangle$ is a matrix algebra in the usual sense.
- As the C_i's have mutually disjoint support, $\langle C_1, \ldots, C_5 \rangle$ is closed under SH-multiplication.
- As $C_1^t = C_1$, $C_2^t = C_2$, $C_3^t = C_4$, $C_4^t = C_3$, and $C_5^t = C_5$, it is closed under conjugate-transposition.

Example 2.2 (cont.)

- Finally, as $J_4 = C_1 + C_2 + C_3 + C_4 + C_5$ and $I_4 = C_1 + C_2$, we conclude that $\langle C_1, \ldots, C_5 \rangle$ is a coherent algebra.
- By construction,
  $A \in \langle C_1, \ldots, C_5 \rangle \ \subseteq \ \langle\langle A \rangle\rangle$.
- Hence, $\langle C_1, \ldots, C_5 \rangle = \langle\langle A \rangle\rangle$.

Example 2.3 (Octahedron)



- Now we will proceed with another example of a coherent algebra, using essentially that it is also a centralizer algebra.
- Namely, we start with the octahedron $\mathcal{O}$.

Example 2.3 (cont.)

- Let $g = (2, 3, 4, 5)$, $h = (1, 2, 3)(4, 5, 6)$.
- Check that $g, h \in Aut(\mathcal{O})$.
- Check that $H = \langle g, h \rangle$ is a transitive group of degree 6.
- $|H| = 24$.
- We wish to construct $\mathcal{W} = V(H, [1, 6])$.

Example 2.3 (cont.)

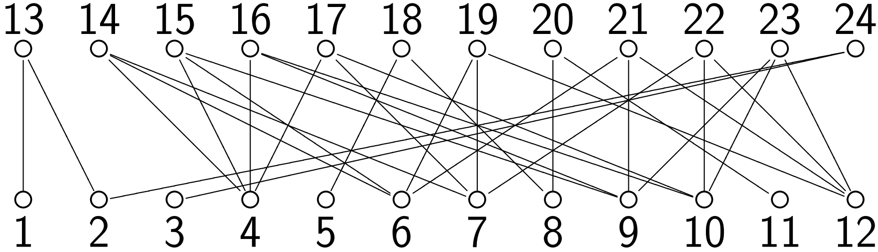
- Construct $\mathcal{W}_1 = V(\langle g \rangle, [1,6])$ and $\mathcal{W}_2 = V(\langle h \rangle, [1,6])$.
- Check that both algebras, by coincidence, have rank 12.
- They are presented by matrices

$$A_1 = \begin{pmatrix} 1 & 4 & 4 & 4 & 4 & 5 \\ 6 & 2 & 7 & 8 & 9 & 10 \\ 6 & 9 & 2 & 7 & 8 & 10 \\ 6 & 8 & 9 & 2 & 7 & 10 \\ 6 & 7 & 8 & 9 & 2 & 10 \\ 11 & 12 & 12 & 12 & 12 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 13 & 14 & 15 & 16 & 17 & 18 \\ 15 & 13 & 14 & 18 & 16 & 17 \\ 14 & 15 & 13 & 17 & 18 & 16 \\ 19 & 20 & 21 & 24 & 22 & 23 \\ 21 & 19 & 20 & 23 & 24 & 22 \\ 20 & 21 & 19 & 22 & 23 & 24 \end{pmatrix}$$

Example 2.3 (cont.)

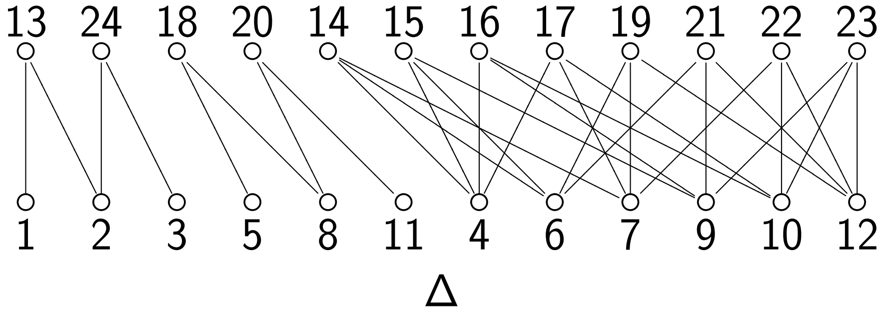
- Now $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ is the intersection of algebras $\mathcal{W}_1$ and $\mathcal{W}_2$.
- To get description of $\mathcal{W}$ consider auxiliary bipartite graph Δ, vertices of which correspond to entries of A_1 and A_2 ($12+12$ vertices).
- Two vertices x, y are adjacent if there exists a cell occupied in A_1 and A_2 by x and y respectively.

Example 2.3 (cont.)



Δ

Example 2.3 (cont.)



Example 2.3 (cont.)

- Connectivity components of the graph Δ define matrix for color graph $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$.
- It has rank 3.
- Different colors correspond to vertices (loops), edges and non-edges of $\mathcal{O}$.
- $A = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 & 2 \\ 2 & 2 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 & 2 & 2 \\ 2 & 2 & 3 & 2 & 1 & 2 \\ 3 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}.$

Schurian coherent algebras

- We call a coherent algebra $\mathcal{W}$ Schurian if it coincides with the centralizer algebra of a suitable permutation group.
- In the previous example, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}$ are all Schurian algebras of ranks 12, 12 and 3 respectively.
- Otherwise, $\mathcal{W}$ is called non-Schurian.

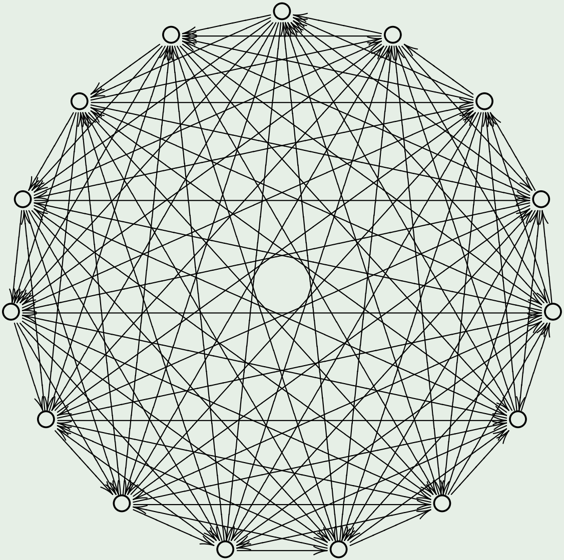
I. Schur

- The name goes back to Schur, who was considering (1933) Schur rings, a special kind of coherent algebras which are simultaneously group algebras (rings).
- Schur believed that all Schur rings are coming from a centralizer algebra of a suitable permutation group.

Non-Schurian

- The smallest counterexamples to the conjecture of Schur in wider context, non-Schurian association schemes, exist on 15, 16, 18 vertices.
- The desired property may be established by group-theoretical or combinatorial arguments.
- In any case this it quite a routine activity (computer is very helpful).

Example 2.4 (DRT on 15 vertices)

$\Gamma =$ 

Example 2.4 (cont.)

- Doubly regular tournament Γ, its opposite graph Γ^t and the reflexive relation form coherent configuration of rank 3.
- The parameters are $(15, 7, 3, 4)$, that is $A(\Gamma)^2 = 4A(\Gamma) + 3A(\Gamma)^t$
- It is non-Schurian: $Aut(\Gamma)$ has order 21, while Γ has $\frac{15 \cdot 14}{2} = 105$ arcs.

Example 2.4 (cont.)

- Alternative (combinatorial) proof: count the number of induced subgraphs with 5 vertices of prescribed isomorphism types;
- Distinguish arcs of Γ, using these invariants.

Main references

- Klin, Mikhail; Rücker, Christoph; Rücker, Gerta; Tinhofer, Gottfried. Algebraic combinatorics in mathematical chemistry. Methods and algorithms. I. Permutation groups and coherent (cellular) algebras. MATCH No. 40 (1999), 7–138.

- Pasechnik, Dmitrii V. Skew-symmetric association schemes with two classes and strongly regular graphs of type $L_{2n-1}(4n-1)$. Interactions between algebra and combinatorics. Acta Appl. Math. 29 (1992), no. 1-2, 129–138.

Thank You!