Jordan schemes

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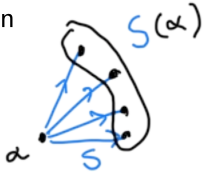
Summer Seminar on Symmetry, Computer Algebra, Algebraic Graph Theory and beyond
Beer-Sheva,
Israel,
September 2025

Binary relations

Let $R, S \subseteq \Omega^2 = \Omega \times \Omega$ be binary relations. Then

- $S^\top := \{(\alpha, \beta) \mid (\beta, \alpha) \in S\}$;
- S is symmetric if $S = S^\top$;
- $S(\alpha) := \{\beta \mid (\alpha, \beta) \in S\}$;
- $RS = \{(\alpha, \beta) \mid R(\alpha) \cap S^\top(\beta) \neq \emptyset\}$;
- $1_\Omega := \{(\omega, \omega) \mid \omega \in \Omega\}$



Matrices

Let $A, B \in M_\Omega(\mathbb{F}), \mathrm{char}(\mathbb{F}) \neq 2$ be arbitrary matrices. We denote by

- AB (or $A \cdot B$) the usual matrix product; e
- $A \circ B$ the Schur-Hadamard (component-wise) product, i.e. $(A \circ B)_{\alpha\beta} := A_{\alpha\beta} B_{\alpha\beta}$;
- $A \star B := \frac{1}{2}(AB + BA)$ the Jordan product of matrices;
- $A^\top$ the transposed of A;
- I_Ω the identity matrix (the $\cdot, \star$-unit);
- J_Ω the all one matrix (the $\circ$-unit);
- if R is a binary relation, then $\underline{R}$ denotes the adjacency matrix of R.

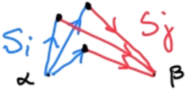
Association schemes (Bose and Shimamoto, 1952).

A set $A_0, \ldots, A_{r-1} \in M_\Omega(\mathbb{Q})$ of $0, 1$-matrices determines a symmetric association scheme $\iff$

1. $I_\Omega = A_0$;
2. $A_i^\top = A_i$;
3. $\sum_i A_i = J_\Omega$;
4. $A_i A_j = \sum_k p_{i,j}^k A_k$.

It follows from the definition that

1. $\mathcal{A} := \langle A_0, \ldots, A_{r-1} \rangle$ is a commutative subalgebra of $M_\Omega(\mathbb{Q})$;
2. $\forall i \; A_i = \underline{S}_i$ for a unique symmetric relation $S_i \subseteq \Omega^2$;
3. the relations $S_0 = 1_\Omega, S_2, \ldots, S_{r-1}$ form a partition of Ω^2;
4. $\forall_{i,j,k} \forall_{(\alpha,\beta) \in S_k} : |S_i(\alpha) \cap S_j(\beta)| = p_{i,j}^k$.



Symmetric association schemes

The numbers r and $|\Omega|$ are called the rank and order of an AS $\mathfrak{X} = (\Omega, \{S_0, ..., S_{r-1}\})$.

The matrices A_i, the corresponding relations S_i and their graphs (Ω, S_i) are called basic matrices/relations/graphs of the scheme.

The numbers $p_{i,j}^k$ are called intersection numbers of $\mathfrak{X}$. They are structure constants of the adjacency algebra $\mathcal{A} = \langle A_0, ..., A_{r-1}\rangle$ w.r.t. standard basis $A_0, ..., A_{r-1}$.

Let $\mathbf{x} = (x_0, ..., x_{r-1})$ be a vector of non-commutative variables. The matrix $A(\mathbf{x}) := \sum_{i=0}^{r-1} x_i A_i$ is called the adjacency matrix of $\mathfrak{X}$. Then

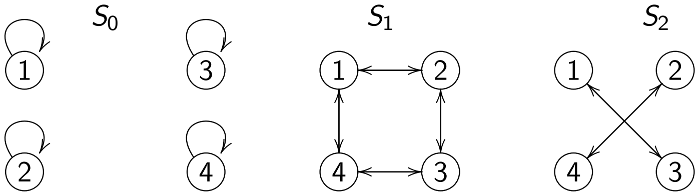
$A(\mathbf{x})^2 = \sum_{i=0}^{r-1} q_i(\mathbf{x}) A_i$ where each $q_i(\mathbf{x})$ is a quadratic form.

A concrete example

$\Omega = \{1, 2, 3, 4\}, r = 3.$

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_1^2 = 2A_0 + 2A_2, A_1 A_2 = A_2 A_1 = A_1, A_2^2 = A_0.$$



A concrete example

Adjacency matrix

$$A(\mathbf{x}) = \begin{pmatrix} x_0 & x_1 & x_2 & x_1 \\ x_1 & x_0 & x_1 & x_2 \\ x_2 & x_1 & x_0 & x_1 \\ x_1 & x_2 & x_1 & x_0 \end{pmatrix} \Rightarrow A(\mathbf{x})^2 = q_0(\mathbf{x})A_0 + q_1(\mathbf{x})A_1 q_2(\mathbf{x})A_2$$

where

$$\begin{aligned} q_0(\mathbf{x}) &= x_0^2 + 2x_1^2 + x_2^2; \\ q_1(\mathbf{x}) &= x_0x_1 + x_1x_0 + x_1x_2 + x_2x_1; \\ q_2(\mathbf{x}) &= x_0x_2 + x_2x_0 + 2x_1^2. \end{aligned}$$

Association schemes

In 1984 Bannai and Ito published a book "Algebraic Combinatorics I; Association schemes" where they developed a comprehensive theory of ASs. They changed the definition of an AS as follows

1. $I_\Omega = A_0$;
2. $A_i^\top \in \{A_0, \dots, A_{r-1}\}$;
3. $\sum_i A_i = J_\Omega$;
4. $A_j A_i = A_i A_j = \sum_k p_{i,j}^k A_k$;

Later P.-H. Zieschang proposed to remove commutativity, i.e. $A_i A_j \neq A_j A_i$ for some i, j. In what follows an AS is not assumed to be commutative.

Notice that each finite group gives rise to a (may be non-commutative) association scheme.

Examples

1. A trivial scheme $(\Omega, \{1_\Omega, \Omega^2 \setminus 1_\Omega\})$;
2. Association schemes coming from permutation groups (Schurian association schemes);
3. Flag association schemes in finite geometries;
4. Distance regular graphs;
5. Finite groups

Strongly regular graphs

A regular graph $\Gamma = (\Omega, S)$ is called strongly regular iff the number $|S(\alpha) \cap S(\beta)|$ depends only on whether α, β are adjacent in Γ.

Proposition

A graph is strongly regular iff it is a basic graph of a symmetric AS of rank three.

Applications of ASs

1. Statistics of experimental designs;
2. Combinatorial design theory;
3. Finite geometry;
4. Permutation groups;
5. Coding theory;
6. Graph isomorphism problem

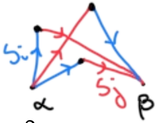
Symmetric Jordan schemes

Bose and Mesner proposed matrix definition of AS in 1959. In the same year Shah generalized the concept of AS and introduced an object later called a (symmetric) Jordan scheme (Cameron).
A set $A_0, \dots, A_{r-1} \in M_\Omega(\mathbb{Q})$ of $0,1$-matrices determines a symmetric Jordan scheme $\iff$

1. $I_\Omega = A_0$;
2. $A_i^\top = A_i$;
3. $\sum_i A_i = J_\Omega$;
4. $A_i \star A_j = \sum_k p_{i,j}^k A_k$;

It follows from the definition that

1. $\mathcal{A} := \langle A_0, \dots, A_{r-1} \rangle$ is a $\star$ subalgebra of $M_\Omega(\mathbb{Q})$;
2. $\forall i \; A_i = \underline{S}_i$ for a unique symmetric relation $S_i \subseteq \Omega^2$;
3. the relations $S_0 = 1_\Omega, S_2, \dots, S_{r-1}$ form a partition of Ω^2;
4. $\forall_{i,j,k} \forall_{(\alpha,\beta)\in S_k} : |S_i(\alpha) \cap S_j(\beta)| + |S_j(\alpha) \cap S_i(\beta)| = 2p_{i,j}^k$



Non-symmetric Jordan schemes

A set $A_0, ..., A_{r-1} \in M_\Omega(\mathbb{Q})$ of $0,1$-matrices determines a non-symmetric Jordan scheme $\iff$

1. $I_\Omega = A_0$;
2. $A_i^\top \in \{A_0, ..., A_{r-1}\}$;
3. $\sum_i A_i = J_\Omega$;
4. $A_i \star A_j = \sum_k p_{i,j}^k A_k$;

It follows from the definition that

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Association schemes and Jordan schemes

Proposition

An association scheme is always a Jordan scheme.

Association schemes and Jordan schemes

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The converse is not true. Take the following partition of $\{1,2\}$: $S_0 = I_{\{1,2\}}, S_1 = \{(1,2)\}, S_2 = S_1^\top$. Its standard basis

$$A_0 = I_2, A_1 = E_{12}, A_2 = E_{21}.$$

The products are

$$A_0^2 = A_0, A_1^2 = A_2^2 = O, A_0 \star A_i = A_i, A_1 \star A_2 = \frac{1}{2} A_0$$

Thus $(\{1,2\}, \{S_0, S_1, S_2\})$ is a non-symmetric (and non-regular) Jordan scheme.

Theorem (MK, MM, SR)

A symmetric Jordan scheme is always regular.

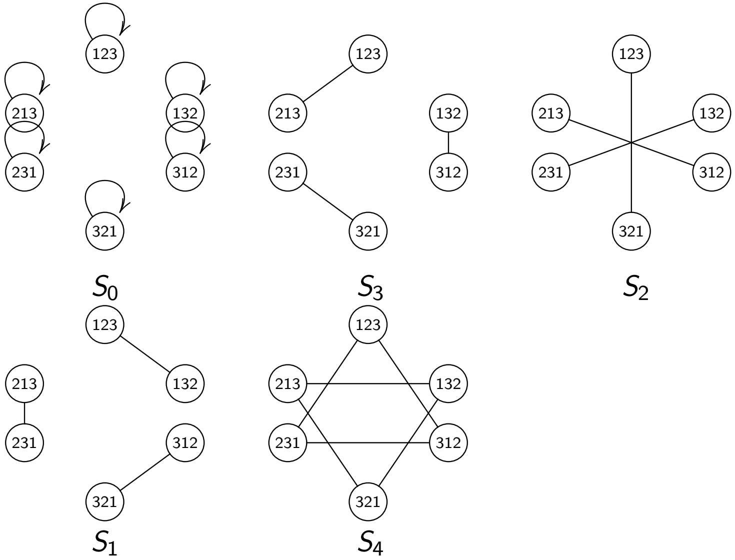
Shah's example

1. $\Omega = \mathrm{Sym}(3)$;
2. $(\pi, \sigma) \in S_0 \iff \pi = \sigma$;
3. $\pi \neq \sigma \ \wedge \ (\pi, \sigma) \in S_i \iff \pi(i) = \sigma(i), i = 1, 2, 3$;
4. $(\pi, \sigma) \in S_4 \iff \forall i : \ \pi(i) \neq \sigma(i)$;

Denote $A_i := \underline{S}_i, A := A_1 + A_2 + A_3$. Then

$$2A_i \star A_j = \begin{cases} A_j & i = 0 \\ A_4 & i, j \in \{1, 2, 3\} \\ A - A_i & i \in \{1, 2, 3\}, j = 4 \\ 2A_0 + A_4 & i = j \end{cases}$$

Basic graphs of Shah's example



An association scheme behind the Shah's example

Let $\Omega = \mathrm{Sym}(3)$.
For each $\sigma \in \mathrm{Sym}(3)$ define $R_\sigma = \{(\pi_1, \pi_2) \mid \pi_1^{-1}\pi_2 = \sigma\}$.

Proposition

The pair $(\Omega, \{R_\sigma\}_{\sigma \in \mathrm{Sym}(3)})$ is a non-commutative association scheme. More precisely,

1. $\underline{R}_\rho^\top = \underline{R}_{\rho^{-1}}$;
2. $\underline{R}_\rho \cdot \underline{R}_\sigma = \underline{R}_{\rho\sigma}$;
3. $S_0 = R_1, S_1 = R_{(23)}, S_2 = R_{(13)}, S_3 = R_{(12)}, S_4 = R_{(123)} + R_{(321)}$

Definition

Given a set $\mathcal{R}$ of binary relations, its symmetrization $\tilde{\mathcal{R}}$ is defined as $\{R^\top \cup R \mid R \in \mathcal{R}\}$.

Symmetrization of an AS

Proposition (R. Bailey)

If $(\Omega, \mathcal{R})$ is an AS then its symmetrization $(\Omega, \tilde{\mathcal{R}})$ is a symmetric Jordan scheme.

A symmetric Jordan scheme is called proper if it is not a symmetrization of an AS.

Question (P. Cameron, 2001)

Are there proper symmetric JS?

Jordan algebras

The product $\star$ is commutative, but not associative. It satisfies the following identity:

$$(A \star B) \star (A \star A) = A \star (B \star (A \star A)).$$

Commutative algebras satisfying the above identity are known as Jordan algebras.

Given an arbitrary associative algebra $(\mathcal{A}, \cdot)$ defined over $\mathbb{F}, \mathrm{char}(\mathbb{F}) \neq 2$, one can define a Jordan product on $\mathcal{A}$ via $A \star B = \frac{1}{2}(AB + BA)$. Then $(\mathcal{A}, \star)$ is a Jordan algebra.

Proposition

A subalgebra of $(\mathcal{A}, \star)$ generated by one element $A \in \mathcal{A}$ coincides with the $\cdot$-subalgebra generated by the same element.

Corollary

Any basic graph of a symmetric Jordan scheme is walk regular $\implies$ it is regular.

Symmetric JSs of small rank

Proposition

Let $\mathfrak{X} = (\Omega, \mathcal{R} = \{R_0, ..., R_{r-1}\})$ be a JS. If $r \leq 4$, then $\mathfrak{X}$ is an AS.

Proof. Consider $\mathcal{A} := \mathbb{R}\langle A_0, A_1, ..., A_{r-1}\rangle$. If $\dim(\mathcal{A}) = r \leq 2$, then $\mathcal{R}$ is trivial.

If degree of the minimal polynomial of some basic graph $(\Omega, R_i), R_i \in \mathcal{R}$ equals to $\dim(\mathcal{A})$, then the adjacency matrix $\underline{R}_i$ $\cdot$-generates $\mathcal{A}$. In this case all matrices in $\mathcal{A}$ pairwise $\cdot$-commute $\Rightarrow \mathcal{R}$ is an AS.

Otherwise, any basic graph of $\mathcal{R}$ has at most 3 eigenvalues. Hence $\mathcal{R}$ is a partition of K_Ω into a disjoint union of two or three SRGs $\Rightarrow \mathcal{R}$ is an AS.

■

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∎

Theorem (MM & van Dam)

Any partition of a complete graph into a disjoint union of three SRGs form an AS.

Sufficient condition to be a Jordan scheme

Proposition

A subspace $\mathcal{A}$ of (symmetric) matrices is $\star$-closed iff it is square closed, i.e. $\forall A \in \mathcal{A} : A^2 \in \mathcal{A}$.

3+1 Lemma

Let $\mathcal{S} = \{1_\Omega, C, S_1, S_2, S_3\}$ be a symmetric regular partition of Ω^2. Assume that for any $i \in \{1, 2, 3\}$ that the partition $\mathcal{S}_i := \{1_\Omega, C, S_i, S \setminus S_i\}$ is an association scheme. Then $\mathcal{S}$ is a Jordan scheme.

Remark. In this situation (Ω, C) is a strongly regular graph (may be disconnected).

A concrete example of a proper Jordan scheme

The main idea: take an improper Jordan scheme and switch some of its colors.

Let $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ be a Galois field $(1 + \alpha + \alpha^2 = 0)$. Consider the natural action of $SL_2(\mathbb{F}_4) \cong A_5$ on the set $\Omega := (\mathbb{F}_4^2)^*$ (the elements of $\mathbb{F}_4^2$ are column vectors).

The action $(SL_2(\mathbb{F}_4), \Omega)$ is faithful and transitive.

Proposition

Two pairs (x, y) and (u, v) belong to the same 2-orbit of $SL_2(\mathbb{F}_4)$ iff one of the following holds

1. $\det(x, y) = \det(u, v) \neq 0$;
2. $\det(x, y) = \det(u, v) = 0$ and there exists α^i s.t. $y = \alpha^i x, v = \alpha^i u$.

The 2-orbits of this action form an association scheme of rank six $\mathcal{S} = \{1_\Omega = C_0, C_1, C_2, S_0, S_1, S_2\}$ where $(u, v) \in C_i \iff v = \alpha^i u$; $(u, v) \in S_i \iff \det(u, v) = \alpha^i$.

Standard basis of the scheme

Denote

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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Then the matrices below form a standard basis of the adjacency algebra of the scheme.

$$\underline{C}_i = \begin{pmatrix} \sigma_i & 0 & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 & 0 \\ 0 & 0 & \sigma_i & 0 & 0 \\ 0 & 0 & 0 & \sigma_i & 0 \\ 0 & 0 & 0 & 0 & \sigma_i \end{pmatrix}, \underline{S}_i = \begin{pmatrix} 0 & \rho_i & \rho_{i+1} & \rho_{i+1} & \rho_i \\ \rho_i & 0 & \rho_i & \rho_{i+1} & \rho_{i+1} \\ \rho_{i+1} & \rho_i & 0 & \rho_i & \rho_{i+1} \\ \rho_{i+1} & \rho_{i+1} & \rho_i & 0 & \rho_i \\ \rho_i & \rho_{i+1} & \rho_{i+1} & \rho_i & 0 \end{pmatrix},$$

The properties of the scheme

1. The multiplication table has the following form:

$$\underline{C}_i \cdot \underline{C}_j = \underline{C}_{i+j},\ \underline{C}_i \cdot \underline{S}_j = \underline{S}_{i+j},\ \underline{S}_i \cdot \underline{C}_j = \underline{S}_{i-j},$$
$$\underline{S}_i \cdot \underline{S}_j = 4\underline{C}_{i-j} + \underline{S}_0 + \underline{S}_1 + \underline{S}_2$$

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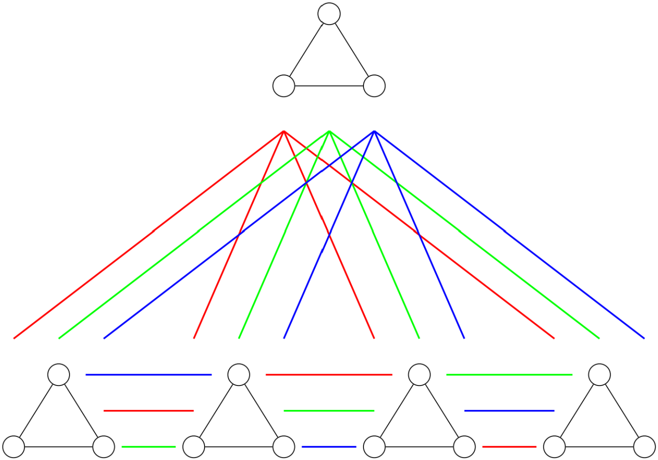
3. Each basic graph (Ω, S_i) is isomorphic to the line graph of the Petersen graph.
4. Each basic graph (Ω, S_i) is distance regular of diameter 3.
5. Each basic graph (Ω, S_i) generates an association scheme of rank 4: $C_0, C_1 \cup C_2, S_i, S \setminus S_i$ where $S = S_0 \cup S_1 \cup S_2$.

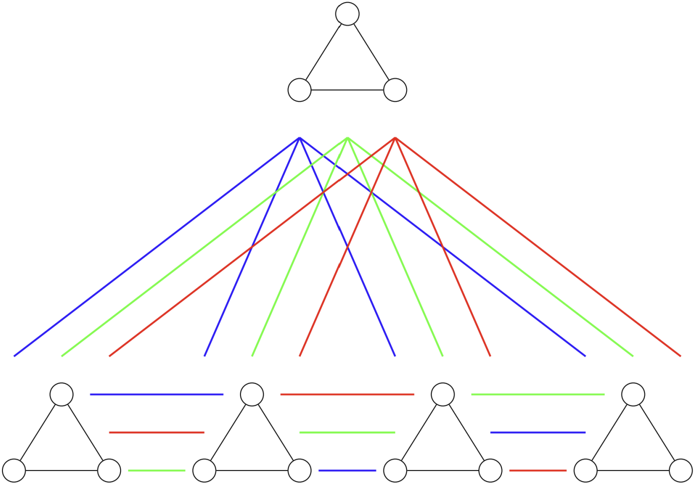
Symmetrization and its switching

The symmetrization of the scheme has rank 5:
$C_0, C := C_1 \cup C_2, S_0, S_1, S_2$. It is an improper Jordan scheme. The new JS will be obtained from it by switching.



Symmetrization and its switching

The switched coloring has the following form:
$C_0, C := C_1 \cup C_2, {\color{green}S_0}, {\color{red}S_1'}, {\color{blue}S_2'}$.



Symmetrization and its switching (in a matrix form)

$$\underline{S}_1 = \begin{pmatrix} 0 & \rho_1 & \rho_2 & \rho_2 & \rho_1 \\ \rho_1 & 0 & \rho_1 & \rho_2 & \rho_2 \\ \rho_2 & \rho_1 & 0 & \rho_1 & \rho_2 \\ \rho_2 & \rho_2 & \rho_1 & 0 & \rho_1 \\ \rho_1 & \rho_2 & \rho_2 & \rho_1 & 0 \end{pmatrix}, \underline{S}_2 = \begin{pmatrix} 0 & \rho_2 & \rho_0 & \rho_0 & \rho_2 \\ \rho_2 & 0 & \rho_2 & \rho_0 & \rho_0 \\ \rho_0 & \rho_2 & 0 & \rho_2 & \rho_0 \\ \rho_0 & \rho_0 & \rho_2 & 0 & \rho_2 \\ \rho_2 & \rho_0 & \rho_0 & \rho_2 & 0 \end{pmatrix}$$

$$\underline{S'}_1 = \left(\begin{array}{c|cccc} 0 & \rho_2 & \rho_0 & \rho_0 & \rho_2 \\ \hline \rho_2 & 0 & \rho_1 & \rho_2 & \rho_2 \\ \rho_0 & \rho_1 & 0 & \rho_1 & \rho_2 \\ \rho_0 & \rho_2 & \rho_1 & 0 & \rho_1 \\ \rho_2 & \rho_2 & \rho_2 & \rho_1 & 0 \end{array}\right), \underline{S'}_2 = \left(\begin{array}{c|cccc} 0 & \rho_1 & \rho_2 & \rho_2 & \rho_1 \\ \hline \rho_1 & 0 & \rho_2 & \rho_0 & \rho_0 \\ \rho_2 & \rho_2 & 0 & \rho_2 & \rho_0 \\ \rho_2 & \rho_0 & \rho_2 & 0 & \rho_2 \\ \rho_1 & \rho_0 & \rho_0 & \rho_2 & 0 \end{array}\right)$$

Proper Jordan scheme

Theorem

The partition $\{1_\Omega, C, S_0, S_1', S_2'\}$ is a proper JS on Ω .

Proposition

The graphs (Ω, S_i) and (Ω, S_i') are isomorphic.

Proof. First, note that $\sigma_i\rho_j = \rho_{i+j}, \rho_i\sigma_j = \rho_{i-j}$. Let P be a block diagonal matrix defined as follows $P := \text{diag}(\sigma_1, \sigma_0, \sigma_0, \sigma_0, \sigma_0)$. P is a permutation matrix of order 3. Direct check shows that $PS_1P^{-1} = S_1', P^{-1}S_2P = S_2'$. □

Corollary

The above partition satisfies the assumptions of $3 + 1$ Lemma $\Longrightarrow$ it forms a Jordan scheme.

Coherent and Jordan closures

Definition

A coherent/Jordan closure of a set $\mathcal{A} \subseteq M_\Omega(\mathbb{Q})$, notation $WL(\mathcal{A})/J(\mathcal{A})$, is the smallest subspace of $M_\Omega(\mathbb{Q})$ containing I_Ω, J_Ω an is closed with respect ${}^\top, \circ, \cdot / {}^\top, \circ, \star$.

Proposition

Let $\mathcal{S} = \{S_0 = 1_\Omega, ..., S_{r-1}\}$ be a regular partition of Ω^2 and $\mathcal{A} := \langle \underline{S}_i \rangle_{i=0}^{r-1} \subseteq M_\Omega(\mathbb{Q})$. Then

1. $J(\mathcal{A}) \subseteq WL(\mathcal{A})$;
2. If $\mathcal{A}$ is symmetric, then $J(\mathcal{A})$ is symmetric. In particular, $J(\mathcal{A}) \subseteq \widetilde{WL(\mathcal{A})}$;
3. If $\mathcal{S}$ is a JS, then it is proper iff $\mathcal{A} \neq \widetilde{WL(\mathcal{A})}$

Finishing the proof

The element

$$(\underline{S'}_1 \cdot \underline{S'}_2) \circ \underline{C} = \begin{pmatrix} 4\sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_1 + 3\sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 + 3\sigma_2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 + 3\sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1 + 3\sigma_2 \end{pmatrix}$$

belongs to $WL(\mathcal{A})$ but doesn't belong to $\mathcal{A}$.

Proper Jordan schemes of unbounded rank

Let $\mathfrak{X} := (\Omega, \{C_0, \ldots, C_{m-1}, S_0, \ldots, S_{m-1}\})$ be an association scheme with the following multiplication table:

$$\begin{array}{rcl} \underline{C}_i \cdot \underline{C}_j & = & \underline{C}_{i+j}, \\ \underline{C}_i \cdot \underline{S}_j & = & \underline{S}_{i+j}, \\ \underline{S}_i \cdot \underline{C}_j & = & \underline{S}_{i-j}, \\ \underline{S}_i \cdot \underline{S}_j & = & n\underline{C}_{i-j} + \frac{n-1}{m}(\underline{S}_0 + \ldots + \underline{S}_{m-1}) \end{array}$$

where m, n are positive integers satisfying $m \mid (n-1)$. Association schemes of this type were constructed by Kharaghani and Suda using generalized weighing matrices (2018), and by Reichard using permutation groups (2016).

Proper Jordan schemes of unbounded rank

1. Every C_i is a permutation; $\{C_i\}_{i=0}^{m-1}$ is a group isomorphic to $\mathbb{Z}_m$.
2. The union $C := \bigcup_i C_i$ forms an equivalence relation with $n+1$ classes of size m.
3. Every basic graph (Ω, S_i) is a diameter 3 antipodal DRG which is a cover of K_{n+1}.
4. If $n = 1$, then the scheme is thin; it corresponds to a dihedral group of order $2m$.

Theorem

The symmetrization $\widetilde{\mathfrak{X}}$ allows a switching which is proper Jordan scheme.

Jordan schemes based on Wallis - Fon-Der-Flaass construction

The vector space $V = \mathbb{Z}_3^d$ has $r := \frac{3^d-1}{2}$ hyperplanes.
Point set $\Omega := V \times \{0, 1, ..., r\}$, $\Omega_i := V \times \{i\}$, $v_i := (v, i)$.
Define $C = \{(u_i, v_i) \mid u \neq v, i = 0, ..., r\}$. The graph (Ω, C) is a disjoint union of $r+1$ complete graphs K_{3^d}.
Using WFDF construction we build three pairwise edge disjoint SRGs $(\Omega, S_0), (\Omega, S_1), (\Omega, S_2)$ with parameters

$$\left(3^d \frac{3^d+1}{2}, 3^{d-1}\frac{3^d-1}{2}, 3^{d-1}\frac{3^{d-1}-1}{2}, 3^{d-1}\frac{3^{d-1}-1}{2}\right)$$

such that

1. $\{1_\Omega, C, S_1, S_2, S_3\}$ is a symmetric regular partition;
2. The partition $\Omega_0, ..., \Omega_r$ is a Hoffman coloring for each of the SRGs (Ω, S_i).

Jordan schemes based on Wallis - Fon-Der-Flaass construction

Proposition

The above partition satisfies the assumptions of 3+1 Lemma.

Proof. By Haemers-Tonchev Theorem the rainbow $((\Omega, \{1_\Omega, C, S_i, (S_1 \cup S_2 \cup S_3) \setminus S_i\})$ is an AS. ■

Theorem

Let $\mathfrak{X} = (\Omega, \{1_\Omega, C, S_1, S_2, S_3\})$ be an arbitrary rank five symmetric Jordan scheme of order $3^d \frac{3^d+1}{2}$ and valencies $1, 3^d - 1, 3^{d-1}\frac{3^d-1}{2}, 3^{d-1}\frac{3^d-1}{2}, 3^{d-1}\frac{3^d-1}{2}$ where d is an even integer. Assume that the basic graph (Ω, S) is a disjoint union of complete graphs. Then the scheme is proper.

References

[1] R. C. Bose, Dale M. Mesner. On linear associative algebras corresponding to association schemes of partially balanced designs. Ann. Math. Statist., 30 (1959), 21-38.

[2] B. V. Shah, A generalisation of partially balanced incomplete block designs, Ann. Math. Statist., 30 (1959), 1041-1050.

[3] R. A. Bailey. Association Schemes. Designed Experiments, Algebra and Combinatorics, v. 84 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2004.

[4] P. J. Cameron. Coherent configurations, association schemes and permutation groups. In Groups, combinatorics & geometry (Durham, 2001), pages 55-71. World Sci. Publ., River Edge, NJ, 2003.

References

[5] M. Klin, S. Reichard, A. Woldar, Siamese objects, and their relation to color graphs, association schemes and Steiner designs, Bull. Belg. Math. Soc. 12 (2005) pp. 845-857.

[6] H. Kharaghani, R. Torabi, On decomposition of complete graphs, Graphs and Combinatorics 19(2003), 519-526.

[7] H. Kharaghani, S. Suda, Non-commutative association schemes and their fusion association schemes, Finite Fields and Their Applications 52(2018), 108-125.

[8] S. Reichard, Tatra schemes and their mergings, in G. A. Jones et al. (eds.), Isomorphisms, Symmetry and Computations in Algebraic Graph Theory, Selected papers based on the presentations at the workshop on algebraic graph theory, Pilsen, Czech Republic, October 3-7, 2016. Springer Proc. Math. Stat. 305 (2020), 219-234.

References

[9] W.H. Haemers, V.D. Tonchev, Spreads in strongly regular graphs, Designs, Codes and Cryptography, 8 (1996), 144-157.

[10] E.R. van Dam, M. Muzychuk, Some implications on amorphic association schemes, JCT(A), 117 (2010), 111-127.

[11] M. Klin, M. Muzychuk, S. Reichard, Proper Jordan schemes exist. First examples, computer search, patterns of reasoning. An essay, 2019, `https://arxiv.org/pdf/1911.06160.pdf`

[12] M. Muzychuk, S. Reichard, M. Klin, On Jordan schemes, Israel J. Math. 249 (2022), no. 1, 309-342

Thank you!