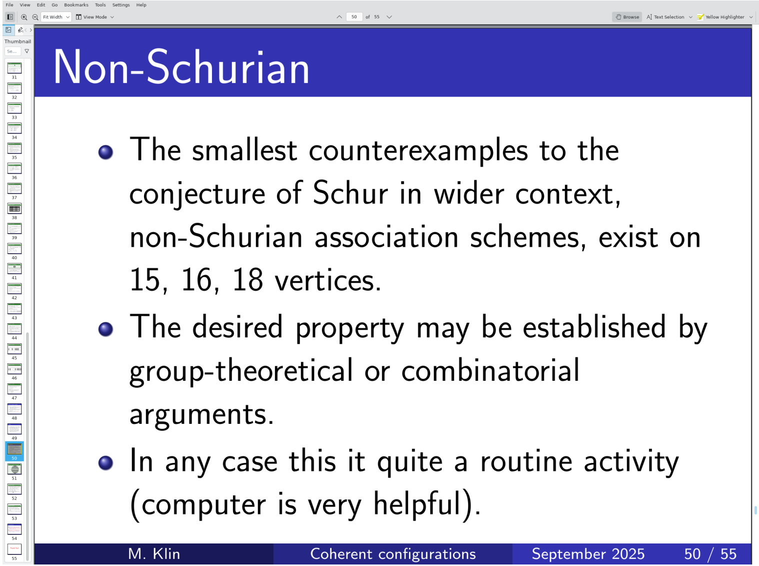
Constructive enumeration of the coherent configurations of order up to 15
Challenges and one surprising result

Matan Ziv-Av (BGU)

September 2025

From MK's second lecture:



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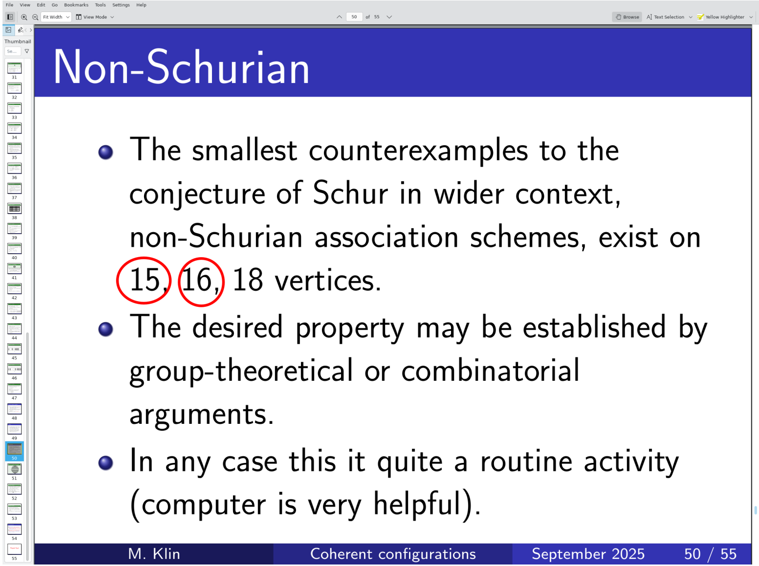
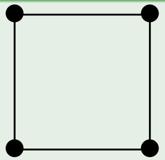


Strongly regular graphs

- A strongly regular graph (SRG) with parameters (v, k, λ, μ) is a regular graph of valency k, in which any two adjacent vertices have λ common neighbors, and any two non-adjacent vertices have μ common neighbors.
- If we denote the adjacency matrix of a graph by A, then an algebraic formulation of the condition is:

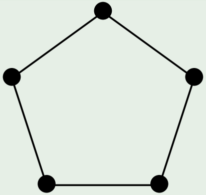
$$A^2 = kI + \lambda A + \mu(J - I - A)$$

Example 1 (Square)



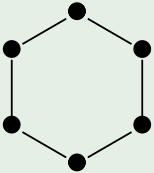
$SRG(4, 2, 0, 2)$

Example 2 (Pentagon)



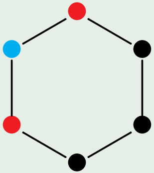
$SRG(5, 2, 0, 1)$

Example 3 (Hexagon)



Not strongly regular.

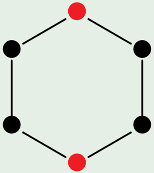
Example 3 (Hexagon)



Not strongly regular.

Some non-neighbours have one common neighbor.

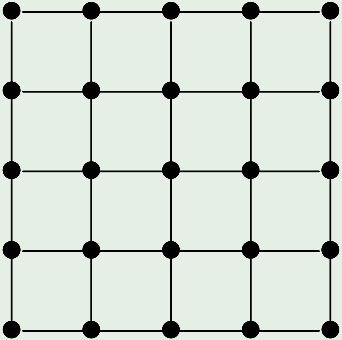
Example 3 (Hexagon)



Not strongly regular.

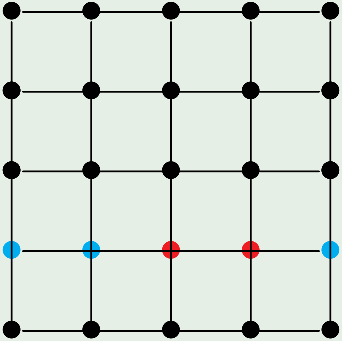
Other non-neighbours have none.

Example 4 (Lattice graph)



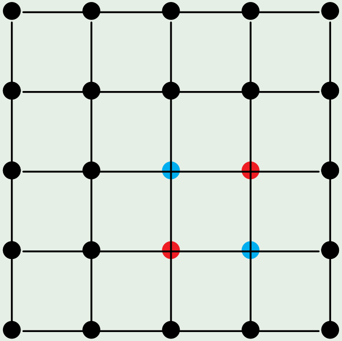
$$SRG(n^2, 2(n-1), n-2, 2)$$

Example 4 (Lattice graph)



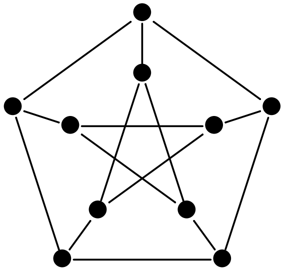
$SRG(n^2, 2(n-1), n-2, 2)$

Example 4 (Lattice graph)



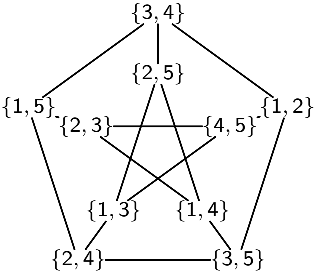
$SRG(n^2, 2(n-1), n-2, 2)$

Example: Petersen graph



$SRG(10, 3, 0, 1)$

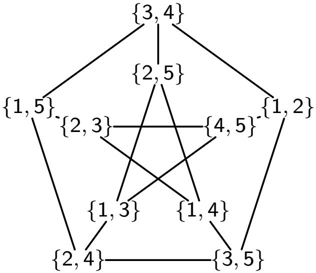
Example: Petersen graph



$SRG(10, 3, 0, 1)$

Easy to see in this model.

Example: Petersen graph



$SRG(10, 3, 0, 1)$

Easy to see in this model.

Unique SRG with those parameters.

Some facts about SRGs

- The complement graph of an SRG with parameters (v, k, λ, μ) is an SRG with parameters $(v, v - k - 1, v - 2 - 2k + \mu, v - 2k + \lambda)$.
- Disconnected SRG: $\mu = 0$, $n \circ K_m$.
- SRG is primitive if both it and its complement connected.
- Therefore an SRG is imprimitive if $\mu = 0$ or $\mu = k$.

Conditions on parameters

- Counting the number of paths of length two, which are not a part of a triangle results in a necessary condition on parameters.
- $(v - k - 1)\mu = k(k - \lambda - 1)$
- Algebraic graph theory considerations give more conditions.

Conditions on parameters

- An SRG has exactly three eigenvalues, k of multiplicity 1 and $r = \frac{1}{2}\left[(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right]$, $s = \frac{1}{2}\left[(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right]$, whose multiplicities:
- $f = \frac{1}{2}\left[(v-1)-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right]$ and $g = \frac{1}{2}\left[(v-1)+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right]$ are integers.

Sample table of feasible parameters

- Parameters that satisfy all those conditions are called feasible parameters.
- The table is taken from `http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html`

	v	k	λ	μ	r^f	s^g	comments
2!	16	6	2	2	2^6	$(-2)^9$	Shrikhande; 4^2 ...
		9	4	6	1^9	$(-3)^6$	OA(4,3); Bilin2x2(2);
!	17	8	3	4	1.562^8	$(-2.562)^8$	Paley(17); 2-graph
!	21	10	3	6	1^{14}	$(-4)^6$	
		10	5	4	3^6	$(-2)^{14}$	T(7)
-	21	10	4	5	1.791^{10}	$(-2.791)^{10}$	Conf
!	25	8	3	2	3^8	$(-2)^{16}$	5^2
		16	9	12	1^{16}	$(-4)^8$	OA(5,4)
15!	25	12	5	6	2^{12}	$(-3)^{12}$	complete enumeration by Paulus; Paley(25); OA(5,3); 2-graph

Algebraic formulation

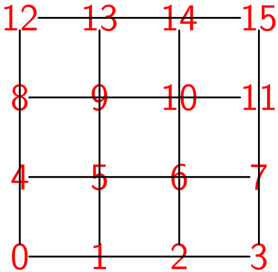
- Let Γ be a (v, k, λ, μ) SRG. We denote its adjacency matrix by A_1, and of its complement by A_2.
- Then, $A_2 = J - I - A_1$
- And together with $A_1^2 = kI + \lambda A_1 + \mu(J - I - A_1)$,
- we get that (I, A_1, A_2) is a symmetric association scheme.
- In other words: $\langle I, A_1, A_2 \rangle$ is a homogeneous symmetric coherent algebra.
- This is another formulation for an SRG. An SRG is a rank 3 symmetric association scheme.

Schurian SRGs

- Some SRGs come from rank 3 transitive groups.
- Those are the Schurian SRGs.
- We are more intereseted in non Schurian ones.
- The smallest of those is or order 16.

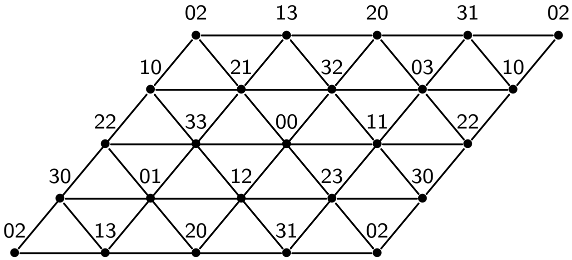
Parameters $(16, 6, 2, 2)$

- Recall that there are two SRGs with parameters $(16, 6, 2, 2)$.
- One of them is the Lattice graph $L_2(4)$



Shrikhande Graph

- The other is the Shrikhande Graph. (Depicted on a torus).



- For two non adjacent vertices. Their two common neighbours, may form an edge (00,10) or a non-edge (00,02).

Not Schurian

- The automorphism group is not transitive on non edges.
- So it is not Schurian.
- This is the smallest non-Schurian SRG.
- This is noted by observing all other SRGS of order 16 or less are Schurian.

Generalization

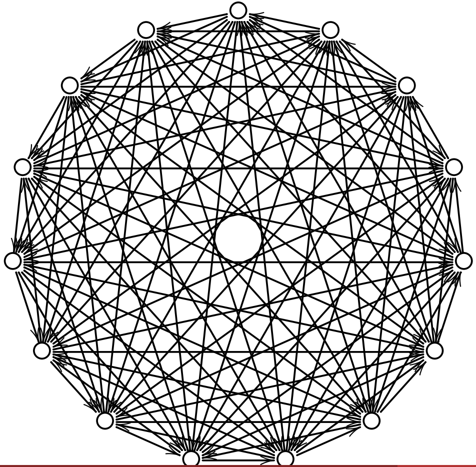
- We saw that it is possible to define a strongly regular graph as a graph whose adjacency matrix A_1, and adjacency matrix of the complement A_2 have the property that the vector space $\langle I, A_1, A_2\rangle$ is a subalgebra of $M_n(\mathbb{C})$.
- We can generalize immediately in two ways: we can allow more than two matrices, and we can allow non symmetric matrices:

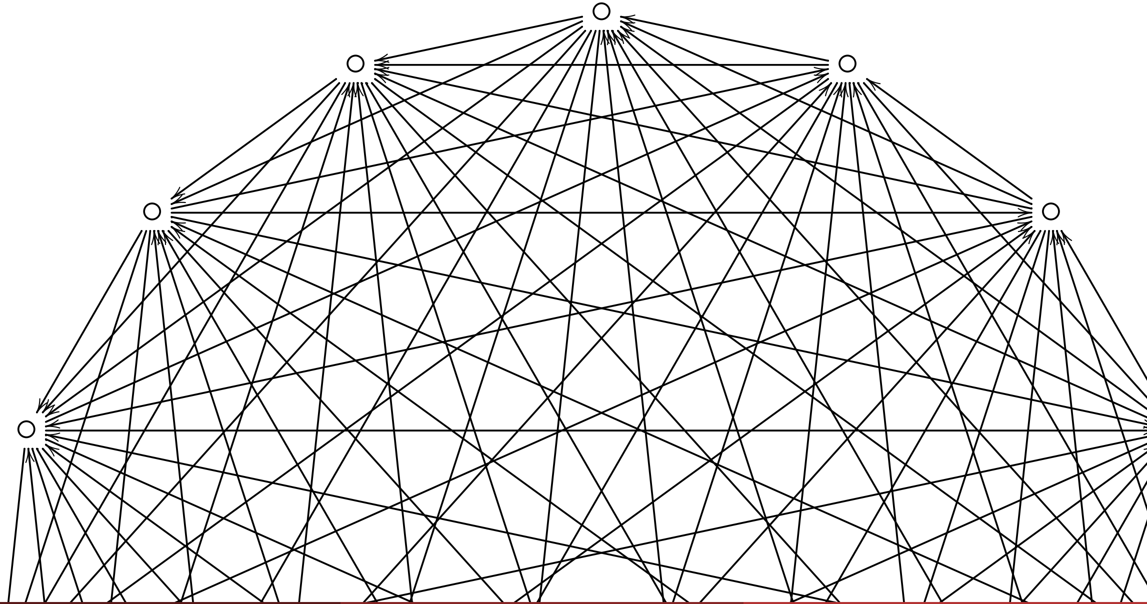
Generalization

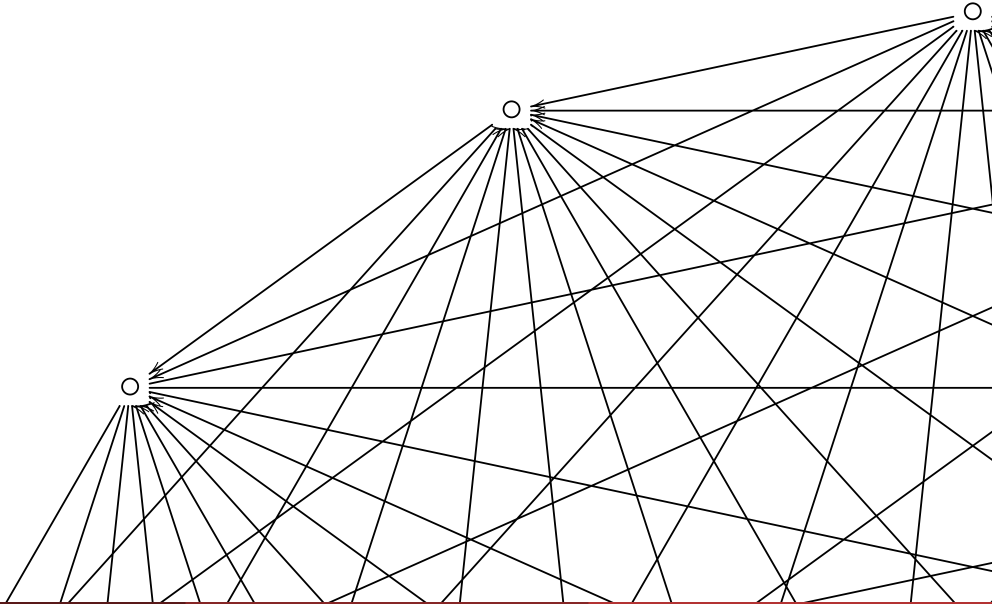
- A homogeneous coherent algebra is a subalgebra of $M_n(\mathbb{C})$ with a basis $(I, A_1, A_2, \ldots, A_r)$, of $0, 1$ matrices, such that $I + A_1 + \cdots + A_r = J$. (And the basis is closed under transposition).
- The tuple of graphs (or relations) corresponding to $(A_1, \ldots, A_r)$ is called an association scheme (AS).

Smallest non Schurian AS

- Recall from MK's second talk: DRT on 15 vertices:







DRT on 15 points

- Doubly regular tournament Γ, its opposite graph Γ^T and the reflexive relation form an association scheme of rank 3 (non symmetric).
- The parameters are $(15, 7, 3, 4)$, that is $A(\Gamma)^2 = 4A(\Gamma) + 3A(\Gamma)^T$
- It is non-Schurian: $Aut(\Gamma)$ has order 21, while Γ has $\frac{15 \cdot 14}{2} = 105$ arcs.

It is non Schurian

- Alternative (combinatorial) proof:
  count the number of induced subgraphs with 5 vertices of prescribed isomorphism types;
- Distinguish arcs of Γ, using these invariants.

- Let X be a finite set and let $\mathcal{R} = \{R_1, R_2, \ldots, R_r\}$.
- If the following conditions hold:

CC1. $R_i \cap R_j = \emptyset$ for $1 \leq i \neq j \leq r$;

CC2. $\cup_{i-1}^{r} R_i = X^2$;

CC3. For each $i \in \{1, 2, \ldots, r\}$ there is an $i' \in \{1, 2, \ldots, r\}$ such that $R_i^t = R_{i'}$;

CC4. There exists a subset $I' \subseteq \{1, 2, \ldots r\}$ such that $\cup_{i \in I'} R_i = \Delta$;

CC5. For each $i, j, k \in \{1, 2, \ldots, r\}$ the number of elements $z \in X$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant provided that $(x, y) \in R_k$. We denote this constant by p_{ij}^k.

- $\mathfrak{M} = (X, \mathcal{R})$ is a coherent configuration.

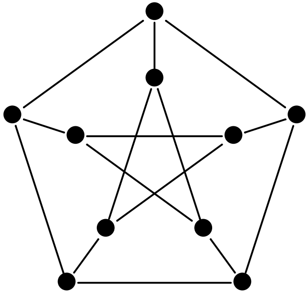
- A source for coherent configurations is 2-orbits of permutation groups.
- If (G, Ω) is a permutation group then the set of orbits of the natural action of G on Ω^2 is a coherent configuration on Ω.
- Denoted by $V(G, \Omega)$.
- Such a coherent configuration is called a Schurian coherent configuration.

- The adjacency matrices of the basic graphs of a coherent configuration form a basis of a matrix algebra.
- Such an algebra is called a coherent algebra.
- A coherent algebra is a subalgebra of $\mathbb{C}^{n\times n}$ that includes the matrices I_n and J_n (the all ones matrix), and is closed under transposition and Schur-Hadamard (element-wise) product.

CC4. There exists a subset $I' \subseteq \{1, 2, \ldots r\}$ such that $\cup_{i \in I'} R_i = \Delta$.

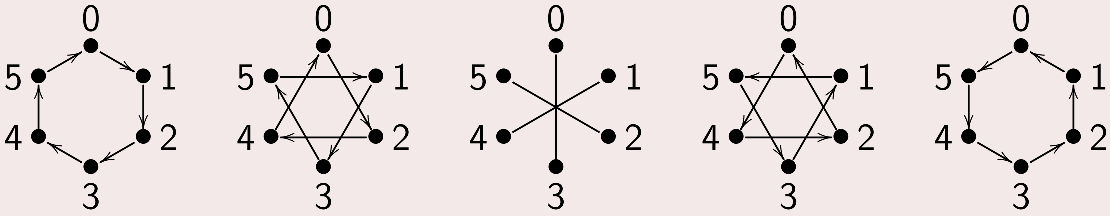
- Together with CC1. we get a partition of Δ, the diagonal of X, $\Delta = \{(x, x) | x \in X\}$.
- This corresponds to a partition $\{f_1, \ldots, f_m\}$ of X.
- f_i is called a fiber of $\mathfrak{M}$.
- A coherent configuration with one fiber is called an association scheme.

- A symmetric association scheme of rank 3 corresponds to a strongly regular graph.
- Schurian association schemes of rank 3 correspond to rank-3 permutation groups.



The Petersen graph – a rank 3 strongly regular graph.

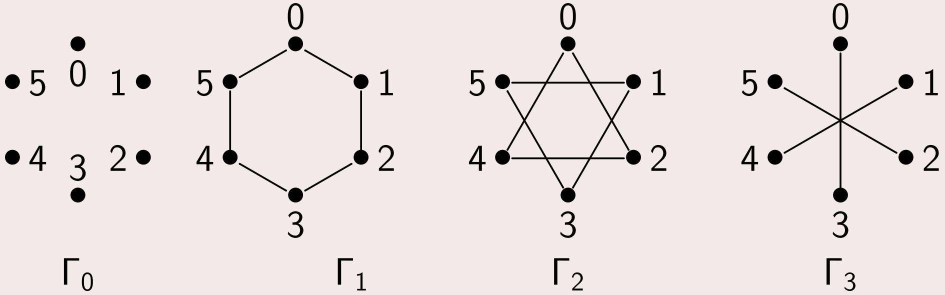
Example 1 ($V(C_6)$)



Γ_1 Γ_2 Γ_3 $\Gamma_4 = \Gamma_2^T$ $\Gamma_5 = \Gamma_1^T$

- *This is an association scheme of rank 6.*
- *If we merge relations* 2 *and* 4, *as well as* 1 *and* 5, *we get a new association scheme.*

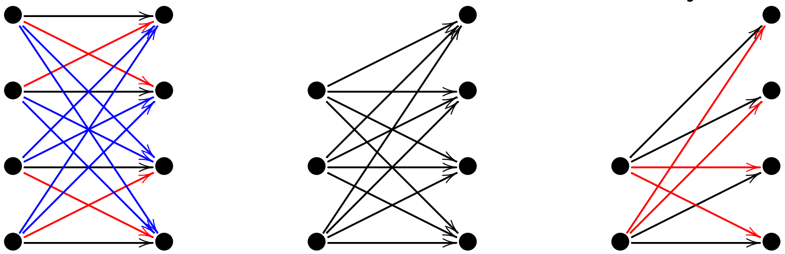
Example 2 ($V(D_6)$)



- *This is an association scheme of rank 4.*
- *Its automorphism group is D_6, the dihedral group of order 12 and degree 6.*

- For each basic relation R_i there exist j, k such that $R_i \subseteq f_j \times f_k$.
- If we take only some of the fibers, $X' = f_{i_1} \cup \cdots \cup f_{i_t}$, we can take only the relevant relations $\mathcal{R}' = \{R \in \mathcal{R} | R \subseteq X' \times X'\}$.
- $\mathfrak{M}' = (X', \mathcal{R}')$ is a coherent configuration. $\mathfrak{M}'$ is called the induced coherent configuration on X'.
- The induced coherent configuration on a single fiber is an association scheme.

- If $R_i \subseteq f_j \times f_k$, $j \neq k$, then we say that R_i is between f_j and f_k.
- A basic graph between two fibers is an oriented biregular (or bipartite semiregular) graph.
- The basic graphs between f_j and f_k form a partition of the complete bipartite directed graph $K_{f_i \to f_j}$.



Example 3 (A two fibers coherent configuration)

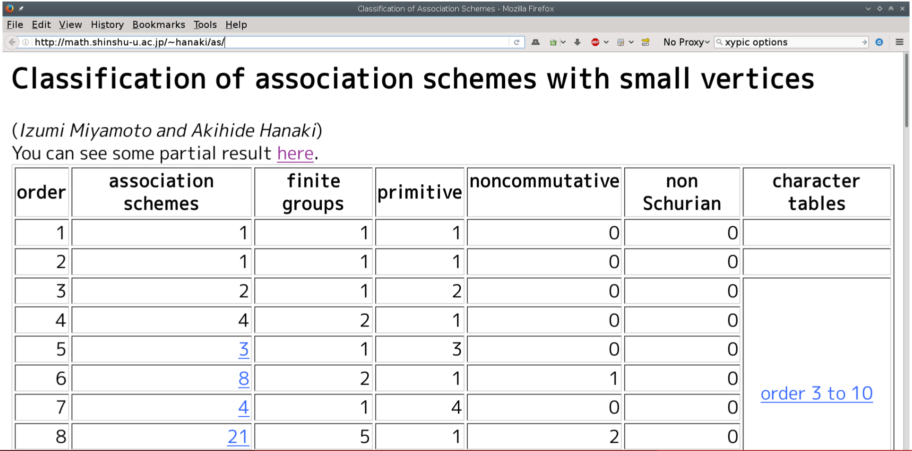
- *We start with an incidence structure with four points and two blocks:* $(\{1,2,3,4\},\{\{1,2\},\{3,4\}\})$.
- *One fiber will consist of the blocks, and the other of the points.*
- *The relations within each fiber will be: loops, point (or block) graph and its complement.*
- *The relations between the fibers will be the incidence relation and its complement.*

Example 3 (cont.)

- *Altogether we have 9 colors.*
- *A compact way to present a coherent configuration is by its color matrix.*

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 0 & 3 & 3 & 2 & 2 \\ 4 & 5 & 6 & 7 & 8 & 8 \\ 4 & 5 & 7 & 6 & 8 & 8 \\ 5 & 4 & 8 & 8 & 6 & 7 \\ 5 & 4 & 8 & 8 & 7 & 6 \end{pmatrix}$$

- Hanaki and Miyamoto enumerated all association schemes of order up to 30.
- Also, those of order 32,33,34 and 38.


Classification of Association Schemes - Mozilla Firefox

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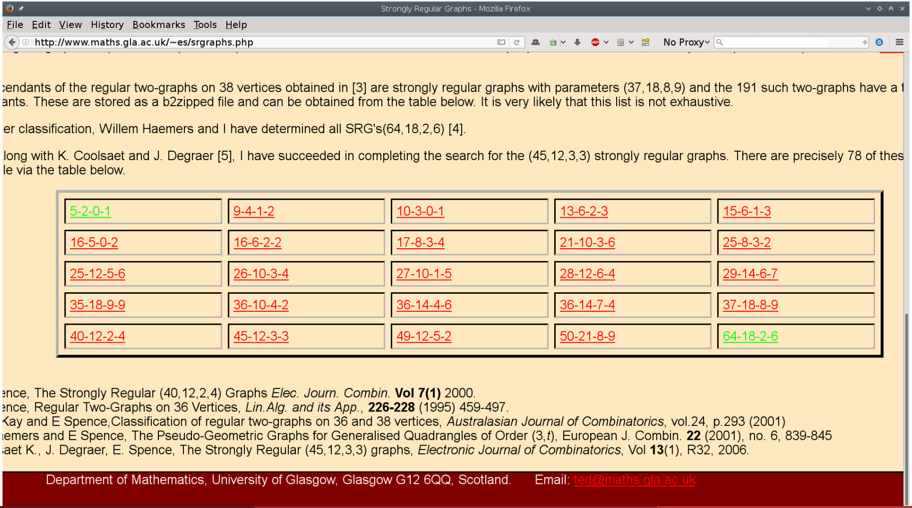
http://math.shinshu-u.ac.jp/~hanaki/as/

Classification of association schemes with small vertices

(*Izumi Miyamoto and Akihide Hanaki*)
You can see some partial result here.

order	association schemes	finite groups	primitive	noncommutative	non Schurian	character tables
1	1	1	1	0	0	
2	1	1	1	0	0	
3	2	1	2	0	0	order 3 to 10
4	4	2	1	0	0	
5	3	1	3	0	0	
6	8	2	1	1	0	
7	4	1	4	0	0	
8	21	5	1	2	0	

- Ted Spence enumerated strongly regular graphs with some parameter sets up to 64 vertices.



cendants of the regular two-graphs on 38 vertices obtained in [3] are strongly regular graphs with parameters (37,18,8,9) and the 191 such two-graphs have a t ants. These are stored as a b2zipped file and can be obtained from the table below. It is very likely that this list is not exhaustive.

er classification, Willem Haemers and I have determined all SRG's(64,18,2,6) [4].

long with K. Coolsaet and J. Degraer [5], I have succeeded in completing the search for the (45,12,3,3) strongly regular graphs. There are precisely 78 of thes le via the table below.

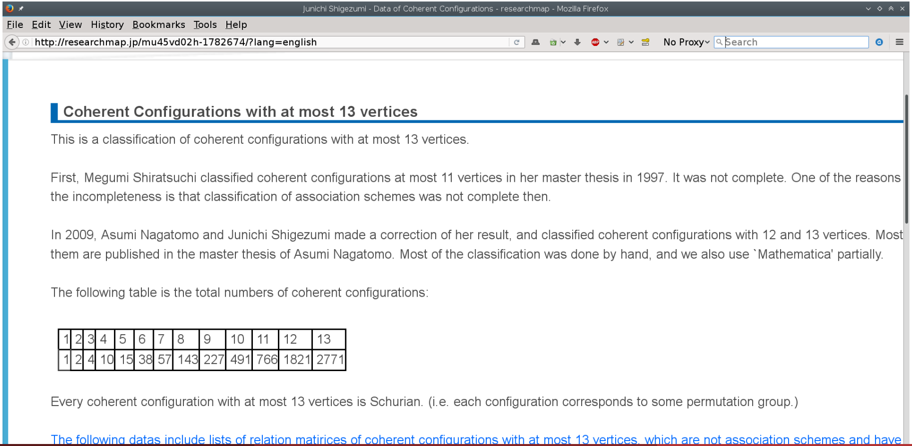
5-2-0-1	9-4-1-2	10-3-0-1	13-6-2-3	15-6-1-3
16-5-0-2	16-6-2-2	17-8-3-4	21-10-3-6	25-8-3-2
25-12-5-6	26-10-3-4	27-10-1-5	28-12-6-4	29-14-6-7
35-18-9-9	36-10-4-2	36-14-4-6	36-14-7-4	37-18-8-9
40-12-2-4	45-12-3-3	49-12-5-2	50-21-8-9	64-18-2-6

ence, The Strongly Regular (40,12,2,4) Graphs *Elec. Journ. Combin.* **Vol 7(1)** 2000.
ence, Regular Two-Graphs on 36 Vertices, *Lin.Alg. and its App.*, **226-228** (1995) 459-497.
Kay and E Spence,Classification of regular two-graphs on 36 and 38 vertices, *Australasian Journal of Combinatorics*, vol.24, p.293 (2001)
emers and E Spence, The Pseudo-Geometric Graphs for Generalised Quadrangles of Order (3,*t*), European J. Combin. **22** (2001), no. 6, 839-845
aet K., J. Degraer, E. Spence, The Strongly Regular (45,12,3,3) graphs, *Electronic Journal of Combinatorics*, Vol **13**(1), R32, 2006.

Department of Mathematics, University of Glasgow, Glasgow G12 6QQ, Scotland. Email: ted@maths.gla.ac.uk

- Sven Reichard and Christian Pech enumerated S-rings over groups of order up to 47.
- MZ extended the results for groups of order up to 63.
- `http://my.svgalib.org/s-rings/wschur.tar.gz`

- Shiratsuchi (1997) and Nagatomo & Shigezumi (2009) enumerated CCs of order up to 13.
- Sven Reichard (2012) enumerated CCs of order up to 13.



Junichi Shigezumi - Data of Coherent Configurations - researchmap - Mozilla Firefox

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Coherent Configurations with at most 13 vertices

This is a classification of coherent configurations with at most 13 vertices.

First, Megumi Shiratsuchi classified coherent configurations at most 11 vertices in her master thesis in 1997. It was not complete. One of the reasons the incompleteness is that classification of association schemes was not complete then.

In 2009, Asumi Nagatomo and Junichi Shigezumi made a correction of her result, and classified coherent configurations with 12 and 13 vertices. Most them are published in the master thesis of Asumi Nagatomo. Most of the classification was done by hand, and we also use `Mathematica' partially.

The following table is the total numbers of coherent configurations:

1	2	3	4	5	6	7	8	9	10	11	12	13
1	2	4	10	15	38	57	143	227	491	766	1821	2771

Every coherent configuration with at most 13 vertices is Schurian. (i.e. each configuration corresponds to some permutation group.)

The following datas include lists of relation matirices of coherent configurations with at most 13 vertices, which are not association schemes and have

- A block matrix representation of a coherent configuration is:

$$\left(\begin{array}{c|c} AS_1 & CB \\ \hline CB^t & AS_2 \end{array}\right) \quad \left(\begin{array}{c|c|c|c} AS_1 & CB_1 & CB_2 & CB_3 \\ \hline CB_1^t & AS_2 & CB_4 & CB_5 \\ \hline CB_2^t & CB_4^t & AS_3 & CB_6 \\ \hline CB_3^t & CB_5^t & CB_6^t & AS_4 \end{array}\right) \quad \left(\begin{array}{ccc|c} & & & CB_1 \\ & CC & & CB_2 \\ & & & CB_3 \\ \hline CB_1^t & \multicolumn{1}{|c|}{CB_2^t} & CB_3^t & AS \end{array}\right)$$

- CB is a color partition of the complete (directed) bipartite graph into biregular subgraphs.
- CB^t is not exactly the transpose of CB.

- Thus the problem reduces into two smaller problems
  - What can be in place of AS_i.
  - What can be in place of CB_i.
- For AS_i, we can use the results of Hanaki and Miyamoto.

- We start by enumerating coherent configurations with two fibers and order n.
- For every pair of association schemes of orders k, $n - k$, we need to find all possible CB.
- Not every partition of the complete bipartite graph works, since the coherency requirement (CC5) might fail.

Example 4

$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ a & 0 & 1 & 2 & 3 & 4 & 3 \\ b & 2 & 0 & 1 & 4 & 3 & 3 \\ c & 1 & 2 & 0 & 3 & 3 & 4 \\ d & 6 & 7 & 6 & 9 & 10 & 10 \\ e & 7 & 6 & 6 & 10 & 9 & 10 \\ f & 6 & 6 & 7 & 10 & 10 & 9 \end{array}$$

- *This is not a coherent configuration.*
- *For $i, j, k = 1, 3, 3$, the only triangle colored $(1, 3, 3)$ that start with a is (a, b, f).*
- *Both (a, d) and (a, f) are in R_3. But for (a, f) there is one z, and for (a, d) there are none.*

- One partition always works, a partition with one cell.
- In this case the coherent configuration is called a direct product of the two association schemes.
- In any biregular graph with n vertices on one side and m vertices on the other, and valencies l, k,

$$nl = mk.$$

- Thus, if the orders of the association schemes are coprime, the only coherent configuration is their direct product.

- If one of the schemes is of order 1, or in other words, one of the fibers is of size 1, the orders are always coprime.
- This means there is only one way to extend a coherent configuration by a fiber of size 1.
- For this reason we do not include such coherent configurations in our constructive enumeration.
- If the number of coherent configurations of order i without fibers of size 1 is a_i, then the number of all coherent configurations of order n is $s_n = \sum_{i=1}^{n} a_i$.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 \\ 2 & 0 & 1 & 4 & 3 & 3 \\ 1 & 2 & 0 & 3 & 3 & 4 \\ 6 & 7 & 6 & 9 & 10 & 10 \\ 7 & 6 & 6 & 10 & 9 & 10 \\ 6 & 6 & 7 & 10 & 10 & 9 \end{pmatrix} \end{array}$$

- What we could learn from the Example 4 is not only that this specific partition is not a coherent configuration.
- We can also conclude that $R_3 = \{(a, d), (a, f), (b, e), (b, f), (c, d), (c, e)\}$ is not a cell of any CB that gives a coherent configuration with those two specific association schemes.

- So the search for CB is done in two parts:
  - Find all biregular relations which may appear as cells in some CB for given AS_1 and AS_2. (Compatible with AS_1 and AS_2.)
  - Construct all partitions which indeed result with a coherent configuration from the possible cells.

- The first step is straightforward.
- We construct all biregular graphs with the given sizes of parts.
- For each graph Γ, we check compatibility with each AS of the corresponding orders:
    - Multiplying Γ with its transpose does not split any relation of AS.
    - Multiplying Γ by any relation of AS does not break Γ.
- In this project this step was implemented using a C program.

- The number of biregular graphs with part sizes m, n and valencies s, t, denoted $B(m, s; n, t)$, was calculated by McKay for some values.
- `https://cs.anu.edu.au/people/Brendan.McKay/data/semiregular.html`
- For a given (even) $m + n$ it is largest when $n = m$.
- An excerpt from the page might explain why this project stops at order 15 for now.

```
Bv[7,1,7,1] = 5040          Bv[8,1,8,1] = 40320
Bv[7,2,7,2] = 3110940       Bv[8,2,8,2] = 187530840
Bv[7,3,7,3] = 68938800      Bv[8,3,8,3] = 24046189440
                            Bv[8,4,8,4] = 116963796250
```

- For the second step, for given two partitions AS_1 we construct all partitions, for each checking if the result is indeed a coherent configuration.
- We can check whether two cells may be in the same CB with each other. We use this to reduce the number of partitions constructed and tested.
- In this step we make use of the knowledge of the (color) automorphism groups of the given association schemes, therefore it was easier to implement it within GAP.
- This step might be optimized by using WL-stabilization.

Example 5 (Coherent configurations of order 4)

- *Let us construct all coherent configurations of order 4.*
- *The only possible sizes of fibers are* $2, 2$. *There is one association scheme of order 2.*
- *The matrix for us to fill is:* $\begin{pmatrix} 0 & 1 & ? & ? \\ 1 & 0 & ? & ? \\ ? & ? & 2 & 3 \\ ? & ? & 3 & 2 \end{pmatrix}$.
- *There are three biregular graphs from* $\{1, 2\}$ *to* $\{3, 4\}$*:* $\{(1, 3), (2, 4)\}$, $\{(1, 4), (2, 3)\}$, $\{(1, 3), (1, 4), (2, 3), (2, 4)\}$.
- *All those pass the tests.*
- *There are two possible partitions.*

Example 5 (cont.)

- *They correspond to two matrices:*

$$\begin{pmatrix} 0 & 4 & 3 & 3 \\ 4 & 0 & 3 & 3 \\ 2 & 2 & 1 & 5 \\ 2 & 2 & 5 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 6 & 5 & 4 \\ 6 & 0 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 2 & 3 & 7 & 1 \end{pmatrix}$$

- *Each is a color graph of a coherent configuration.*
- *Together with 4 association schemes of order 4 there are 6 coherent configurations of order 4 (without fibers of size 1).*

Example 6 (Coherent configurations of order 5)

- *Let us construct all coherent configurations of order 5.*
- *The only possible sizes of fibers are* $2, 3$.
- $\gcd(2,3) = 1$, *so we know in advance that we have only direct products of associations schemes of orders 2 and 3. With one association scheme of order 2 and 2 of order 3 we get*

$$\begin{pmatrix} 0 & 4 & 3 & 3 & 3 \\ 4 & 0 & 3 & 3 & 3 \\ 2 & 2 & 1 & 5 & 5 \\ 2 & 2 & 5 & 1 & 5 \\ 2 & 2 & 5 & 5 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 3 & 3 & 3 \\ 4 & 0 & 3 & 3 & 3 \\ 2 & 2 & 1 & 5 & 6 \\ 2 & 2 & 6 & 1 & 5 \\ 2 & 2 & 5 & 6 & 1 \end{pmatrix}$$

- *Together with 3 association schemes of order 5, there are 5 ("non-trivial") coherent configurations of order 5.*

$$\left(\begin{array}{ccc|c} & CC & & \begin{array}{c} CB_1 \\ \vdots \\ CB_f \end{array} \\ \hline CB_1^t & \cdots & CB_f^t & AS \end{array}\right)$$

- To enumerate all coherent configurations with $f+1$ fibers, we start with a list of coherent configurations with f fibers and add one fiber to each.
- We do not need to find the CB's as in the case of two fibers, since we know that
$$\left(\begin{array}{c|c} AS_i & CB_i \\ \hline CB_i^t & AS \end{array}\right)$$
is a two fiber coherent configuration.

- We should note that while we have a list of all two fiber coherent configurations up to isomorphism, $\left(\begin{array}{c|c} AS_i & CB_i \\ \hline CB_i^t & AS \end{array}\right)$ is not necessarily in our list.
- While every coherent configuration is isomorphic to one with specific representatives of association schemes in the diagonal blocks, it is not true that any induced coherent configuration on two fibers needs to be a prescribed representative.

- An automorphism of one of the association schemes acting on the whole coherent configuration will not change the association scheme, but will change the coherent configuration.

Example 7

$$C_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 2 & 3 \\ b & 1 & 0 & 3 & 2 \\ c & 4 & 5 & 6 & 7 \\ d & 5 & 4 & 7 & 6 \end{array} \qquad C_2 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 3 & 2 \\ b & 1 & 0 & 2 & 3 \\ c & 5 & 4 & 6 & 7 \\ d & 4 & 5 & 7 & 6 \end{array}$$

- *The permutation (a, b) which is an automorphism of the first association scheme, acting on rows and columns (but not labels) of C_1 results with C_2.*
- *Which is isomorphic, but not identical.*

- For larger coherent configurations we can find examples where the resulting coherent configurations are not color isomorphic.

$$\left(\begin{array}{ccc|c} & CC & & \begin{matrix} CB_1 \\ \vdots \\ CB_f \end{matrix} \\ \hline CB_1^t & \cdots & CB_f^t & AS \end{array}\right)$$

- The problem is that while we find an automorphism of AS which will map $\left(\begin{array}{c|c} AS_1 & CB_1 \\ \hline CB_1^t & AS \end{array}\right)$ to a coherent configuration in our list, it will not be the same automorphism as the one for $\left(\begin{array}{c|c} AS_2 & CB_2 \\ \hline CB_2^t & AS \end{array}\right)$.

- Where the words automorphism, isomorphism appear above, we actually refer to the weak (color) variants.
- To get the list of two fiber coherent configurations used as construction blocks for coherent configurations with more fibers, we take the orbit of each coherent configuration $\mathfrak{M}$ under the action of the stabilizer of all association schemes in $CAut(\mathfrak{M})$.
- This stabilizer is the same as $CAut(AS_1) \times CAut(AS_2)$.

Order	1	2	3	4	5	6	7	8
CCs	1	2	4	10	15	38	57	143
Schurian	1	2	4	10	15	38	57	143
Association schemes	1	1	2	4	3	8	4	21

Order	9	10	11	12	13	14	15
CCs	228	492	769	1845	2806	6167	9841
Schurian	228	492	769	1845	2806	6166	9839
Association schemes	12	13	4	59	6	16	25

- The smallest non-Schurian coherent configuration $\mathfrak{m}$ is of order 14. It has two fibers, of sizes 6 and 8, and rank 11.
- Of order 15 there are exactly two non-Schurian coherent configurations: the one above with an added fiber of size 1, and a well known rank 3 association scheme.

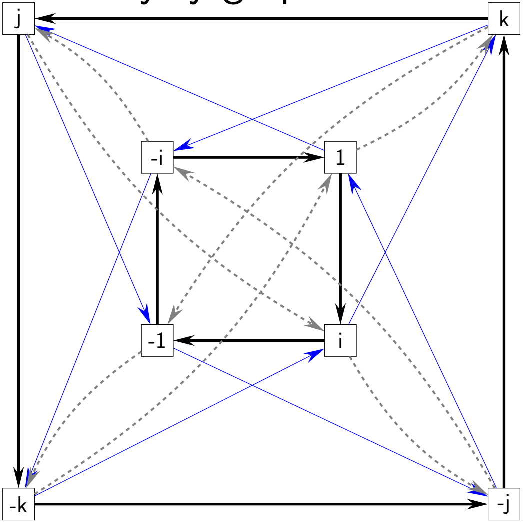
Color matrix of $\mathfrak{m}$

$$\begin{pmatrix}
1 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 4 & 5 & 5 \\
2 & 1 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 4 & 4 & 5 & 4 & 4 \\
3 & 3 & 1 & 2 & 3 & 3 & 4 & 4 & 5 & 4 & 5 & 5 & 5 & 4 \\
3 & 3 & 2 & 1 & 3 & 3 & 5 & 5 & 4 & 5 & 4 & 4 & 4 & 5 \\
3 & 3 & 3 & 3 & 1 & 2 & 4 & 4 & 5 & 5 & 4 & 5 & 4 & 5 \\
3 & 3 & 3 & 3 & 2 & 1 & 5 & 5 & 4 & 4 & 5 & 4 & 5 & 4 \\
6 & 7 & 6 & 7 & 6 & 7 & 8 & 9 & 10 & 10 & 10 & 11 & 11 & 11 \\
6 & 7 & 6 & 7 & 6 & 7 & 9 & 8 & 11 & 11 & 11 & 10 & 10 & 10 \\
6 & 7 & 7 & 6 & 7 & 6 & 11 & 10 & 8 & 10 & 11 & 9 & 10 & 11 \\
7 & 6 & 6 & 7 & 7 & 6 & 11 & 10 & 11 & 8 & 10 & 10 & 11 & 9 \\
7 & 6 & 7 & 6 & 6 & 7 & 11 & 10 & 10 & 11 & 8 & 11 & 9 & 10 \\
6 & 7 & 7 & 6 & 7 & 6 & 10 & 11 & 9 & 11 & 10 & 8 & 11 & 10 \\
7 & 6 & 7 & 6 & 6 & 7 & 10 & 11 & 11 & 10 & 9 & 10 & 8 & 11 \\
7 & 6 & 6 & 7 & 7 & 6 & 10 & 11 & 10 & 9 & 11 & 11 & 10 & 8
\end{pmatrix}$$

- We wish to establish a computer free interpretation in terms friendly for a human.
- Crucial initial fact is that the computer (GAP) told us that $Aut(\mathfrak{m})$ has order 24, orbits of sizes 6 and 8, rank 12, and is isomorphic to $SL(2,3)$.
- Now we are able to proceed with some theoretical reasonings.

- Action of $G = SL(2,3)$ on 8 points is clear: non-zero vectors of $\mathbb{Z}_3^2$.
- G contains just one conjugacy class of groups of order 4: $\mathbb{Z}_4$.
- G contains subgroup Q_8 of quaternions.
- $G \cong \mathrm{Q}_8 \rtimes \mathbb{Z}_3$.
- We are ready to think geometrically: create a Cayley graph $\Gamma = Cay(\mathrm{Q}_8, \{i, j, k\})$.

Cayley graph Γ



- It is easy to check that $Aut(\Gamma)$ coincides with the group G.
- Observation: the graph Γ contains exactly 6 directed quadrangles.

- Note that G was called by Coxeter binary tetrahedral group.
- Our picture is in his spirit (though we did not see it in literature).
- Quotient group $G/\mathbb{Z}_2 \cong A_4$ (origin of the name).
- Let $V = V_1 \cup V_2$, $V_2 = \mathrm{Q}_8$, V_1 is the set of the 6 quadrangles in Γ.

Proposition 1

$Rank(G, V) = 12.$

- **Proof**. Consider cycle index:
- $Z(G) = \frac{1}{24}\left(x_1^{14} + x_1^6 x_2^4 + 6x_1^2 x_2^2 x_4^2 + 8x_1^2 x_3^4 + 8x_2 x_3^2 x_6\right).$
- Construct cycle index of induced action (G, V^2).
- Count the amount of 2-orbits according to CFB lemma.

- Geometrical understanding of the Schurian CC $V(G, V)$ is now available.
- On the set $V_2 = \mathbb{Q}_8$ we have rank 4: Γ, Γ^T, $4 \circ K_2$ and loops.
- Action on V_1, consisting of 6 quadrangles, is A_4 of rank 4.
- From V_1 to V_2 there are two directed relations (incidence and non incidence), and vice versa.
- This is another proof of Proposition 1.

- On each of the two sets V_1 and V_2 there is a pair of opposite directed graphs.
- Merging of each of the two pairs provides Schurian rank 10 CC.
- Merging of the pair in V_2 gives a Schurian rank 11 CC with an automorphism group of order 192.
- Merging of the pair in V_1 gives a non-Schurian rank 11 CC, $\mathfrak{m}$.

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Thank You!