16-vertex graphs with automorphism groups A_4 and A_5 from the icosahedron

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Outline

Abstract

Main MSC 20B25 = Finite automorphism groups of algebraic, geometric, or combinatorial structures (including graph automorphisms).

Additional MCS 05C25 = Graphs and abstract algebra (groups acting on graphs, etc.

The talk addresses a problem in graph representation theory of finite groups - finding vertex-minimal graphs with a given automorphism group. We exhibit two undirected 16-vertex graphs having automorphism groups A_4 and A_5. It improves Babai's bound for A_4 and the graphical regular representation bound for A_5. The graphs are constructed using projectivisation of the vertex-face graph of the icosahedron.

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Introduction — Group representations

Representation theories of groups can be divided into two overlapping parts.

There are group representations which can be called *nonsurjective* such as permutation representations ($G \to \Sigma_X$) and linear representations ($G \to GL(n, k)$) which deal with homomorphisms $G \to Aut(X)$ which are usually not surjective.

Typical problems in this area are classifications — finding all isomorphism classes of homomorphisms (representations/modules) and properties of representations (modules).

A homomorphism $G \to Aut(X)$ extends to a algebra homomorphism $k[G] \to k[Aut(X)]$. Thus group representation theory extends to algebra representation theory.

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Automorphism group theory as part of group representation theory

There are also group representation theories which can be called *bijective representation theories* (such as Euclidean space isometry and graph automorphism theories) which deal with bijective homomorphisms $G \to Aut(X)$ and study full automorphism groups of certain objects.

Examples of problems in this latter area are problems about extremal parameter values for objects having given automorphism groups.

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$f : V \to V$ - graph automorphism provided

$$v \sim w \iff f(v) \sim f(w).$$

The mapping (functor) Aut: **Grph** $\to$ **Grp**, $\Gamma \mapsto Aut(\Gamma)$.

The automorphism group of a graph Γ — $Aut(\Gamma)$, consists of all vertex permutations that preserve adjacency and non-adjacency. This is a subgroup of the symmetric group on the vertex set $\Sigma_{V(\Gamma)}$.

(For quick check: draw the graph, then permute vertices dragging edges along, check if pictures are the same).

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History of theory of graph automorphism groups

Robert Frucht (1939)

Key contribution: Proved the classical Frucht's theorem, establishing that for every finite group G there is a finite undirected graph Γ such that

$$Aut(\Gamma) \simeq G$$

— functor Aut is surjective.

This result established an important link between abstract group theory and graph theory by showing that undirected graph symmetries (using one irreflexive symetric binary relation) can bijectively model any group structure. (In contrast with permutation representations). May help to visualize some group concepts (e.g. direct and semidirect products).

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First nontrivial vertex-minimal graphs: found minimal vertex number for some abelian groups:

Denote by $\mu(G)$ the minimal number of vertices of undirected graphs having automorphism group isomorphic to the group G:

$$\mu(G) = \min_{\Gamma : Aut(\Gamma) \simeq G} |V(\Gamma)|.$$

Trivial cases

$\mu(\Sigma_n) = n$. $D_{2n} = n$.

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Sabidussi, 1958:

- $m = 3, 4, 5 \implies \mu(\mathbb{Z}_m) = 3m$ (not really true for $m = 4$, $\mu(\mathbb{Z}_4) = 10$);
- $\mu(\mathbb{Z}_{p_k^{a_k}}) = 2p_k^{a_k}$, if $p_k^{a_k} \geq 7$ (not really true for $p = 2$, $\mu(\mathbb{Z}_{2^a}) = 2^a + 6$, Harary (exercise) quoting Merriwether, Daugulis, [2]);
- $\mu(\mathbb{Z}_{p_1^{a_1}} \times ... \times \mathbb{Z}_{p_k^{a_k}}) \leq \sum_{i=1}^{k} \mu(\mathbb{Z}_{p_i^{a_i}})$, if a group is a direct product of subgroups having coprime orders, then the disjoint union of vertex minimal graphs for each factor is a graph for this direct product;

History of theory of graph automorphism groups

Classic example - a graph Γ such that $Aut(\Gamma) \simeq \mathbb{Z}_3$.

$Aut(\Gamma)$ is generated by permutation $(123)(456)(789)$.

Orbits: $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$.

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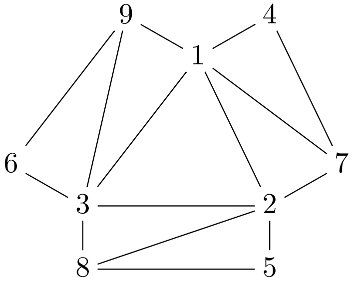

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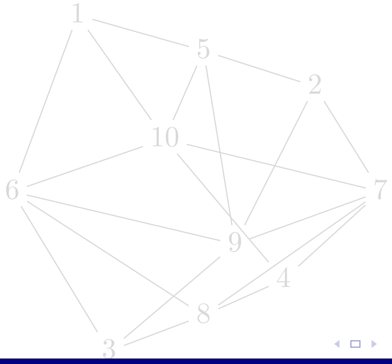
We show a vertex minimal graph Γ such that $Aut(\Gamma) \simeq \mathbb{Z}_4$. $Aut(\Gamma)$ is generated by the vertex permutation $g = (1,3,2,4)(6,9,7,10)(5,8)$.

Subgraphs $\Gamma[6,9,7,10,1,5,2]$ and $\Gamma[6,9,7,10,3,8,4]$ are interchanged by g.



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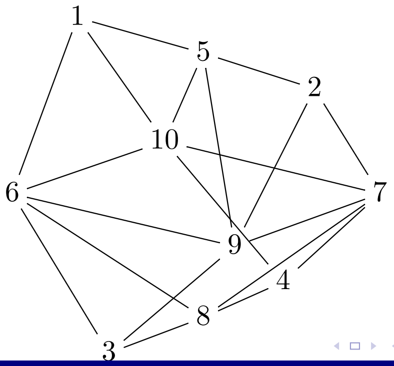

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Laszlo Babai (1974)

An important contribution:
Found a linear upper bound for $\mu(G)$: if $G \neq \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ then

$$\mu(G) \leq 2|G|.$$

Proved by taking two copies of G and encoding a presentation with a minimal set of generators in graph structure (a simpler version for cyclic groups was done earlier, perhaps by G.Sabidussi).

Generalized quaternion groups are examples for which this bound is sharp. There are a few other series of 2-groups with this bound being sharp..

For Σ_*, D_{2*} smaller graphs are known. For most finite groups G, $\mu(G)$ is unknown.

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Some bounds

For $n \geq 5$ A_n admits a graphical regular representation (a graph on $V = G$), see [4]. For these groups $\mu(G) \leq |G|$. Thus for A_5 the best published estimate until 2018 seemed to be $\mu(A_5) \leq 60$.

We exhibit graphs $\Gamma_i = (V, E_i)$, $i \in \{4, 5\}$, such that $|V| = 16$ and

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Γ_4 (also denoted Ξ_I) improves Babai's bound for A_4. Γ_5 (also denoted Π_I) has fewer vertices than the graphical regular representation of A_5.

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Notations

We use standard notations of undirected graphs. A bipartite graph Γ with vertex partition sets V_1 and V_2 is denoted as $\Gamma = (V_1, V_2, E)$.

Given a polyhedron P, we denote its vertex, edge and face sets as $V = V(P)$, $E = E(P)$ and $F = F(P)$, respectively. We can think of P as the triple (V, E, F).

Rotational group of a polyhedron $P = (V, E, F)$ — $Rot(P)$, group of 3D rotations preserving V and E.

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Projectivization

When we think about finding structures having full automorphism group isomorphic to A_5, we may think about the regular icosahedron.

For regular icosahedron I, $Rot(I) \simeq A_5$.

The immediate graph to study is the 1-skeleton graph of I - I_1 (vertex-edge incidence graph, in $V \times E$). But $Aut(I_1) \simeq A_5 \times \mathbb{Z}_2$ - too large.

How to mod out $\mathbb{Z}_2$? One approach - projectivization. (Other possibilities are reflections — "projectivizations" with respect to a line or a plane).

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Definitions of incidence graphs used

If S is a subset of $\mathbb{R}^3$ not containing the origin or a set of vertices/ edges/ faces, then its image under the projectivisation map to $P(\mathbb{R}^3)$ is denoted by $\pi(S)$ or $[S]$.

Definition

Let $P = (V, E, F)$ be a polyhedron. An undirected bipartite graph $\Gamma_P = (V, F, I)$ is the **vertex-face graph of P** if

$$v \sim f \text{ iff } v \in V, f \in F \text{ and } v \in f.$$

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Instead of V, F consider $[V], [F]$, as sets. Join $v_p \in [V]$ and an element of $f_p \in [F]$ iff an inverse image under projectivization of v_p belongs to an inverse image under projectivization of f_p:

$$\pi^{-1}(v_p) \cap \pi^{-1}(f_p) \neq \varnothing.$$

Definition

Let $S = (V, E, F)$ be a centrally symmetric polyhedron. Let S be positioned in $\mathbb{R}^3$ so that its center is at $(0, 0, 0)$. We call the undirected bipartite graph $\Pi_S = ([V], [F], I_p)$ **projective vertex-face graph** if for any $v_p \in [V]$, $f_p \in [F]$ we have $v_p \sim f_p$ iff $v \in f$ for some $v \in \pi^{-1}(v_p)$ and $f \in \pi^{-1}(f_p)$.

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Some cardinalities

$$|[V]| = \frac{|V|}{2}, \; |[E]| = \frac{|E|}{2}, \; |[F]| = \frac{|F|}{2}.$$

Projective vertex-face graph of the icosahedron and A_5

Let $I = (V, E, F)$ be the regular icosahedron. Define $\Gamma_5 := \Pi_I$, it is shown in Fig.1.

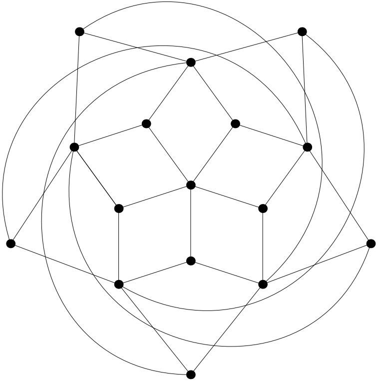

Fig.1. - Π_I.

A_5 theorem

Theorem

Let I be the regular icosahedron. Then $Aut(\Pi_I) \simeq A_5$.

Proof. We prove that $Rot(I) \simeq Aut(\Pi_I)$ in two steps.

First we show that there is a subgroup in $Aut(\Pi_I)$ isomorphic to $Rot(I)$ - the group of rotational symmetries of I. That would mean

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There is an injective group morphism

$$f : Rot(I) \xrightarrow{f_1} Aut(\Gamma_I) \xrightarrow{f_2} Aut(\Pi_I).$$

$f_1 : Rot(I) \to Aut(\Gamma_I)$ maps every $\rho \in Rot(I)$ to $f_1(\rho) \in Aut(\Gamma_I)$ which is the permutation of $V \cup F$ induced by ρ:

$$f_1(\rho)(x) = \rho(x)$$

for any $x \in V \cup F$.

Rotations of I preserve the vertex-face incidence relation, f_1 is a group morphism.

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$f_2 : Aut(\Gamma_I) \to Aut(\Pi_I)$ maps every φ to $\varphi_P := f_2(\varphi)$ defined by the rule

$$\varphi_P([x]) = [\varphi(x)]$$

for any $x \in V(\Gamma_I)$. It is the mapping of projective classes induced by φ. Projectivization and composition commute therefore f_2 is a group morphism.

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All vertices in $[V]$ have degree 5, all vertices in $[F]$ have degree 3. Therefore $[V]$ and $[F]$ are unions of orbits.

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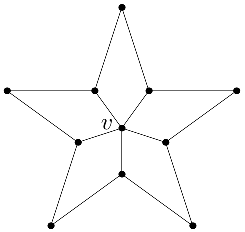
Every vertex class $v \in [V]$ is contained in its 2-neighbourhood — the induced subgraph $\sigma(v)$, shown in Fig.2.



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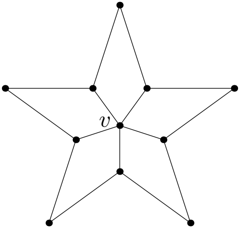

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After mapping v to v' we also have to map isomorphically its 2-neighbourhood $\sigma(v)$ to the 2-neighbourhood of v' — $\sigma(v')$.

The subgraph $\sigma(v)$ has dihedral group D_5 symmetry, it can be automapped in at most 10 ways:

- rotating neighbours of v (in 5 ways),
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2-neighbourhood of any element v, $\sigma(v)$ contains all 6 elements of $[V]$.

Thus a permutation of $[V]$ corresponding to an automorphism can be done in at most $6 \cdot 10 = 60$ ways.

Any permutation of $[V]$ by an automorphism determines a unique permutation of $[F]$.
This is proved by considering again 2-neighbourhoods of $[V]$. If φ_1, φ_2 coincide on V, then $\varphi_1 \varphi_2^{-1}$ fixes all elements of $[V]$, fixes both common neighbours (elements of $[F]$) of two $[V]$ elements in distance 2 and, therefore, all $[F]$ elements.

Thus $|Aut(\Pi_I)| \leq 60$. We have shown that

$$Aut(\Pi_I) = f(Rot(I)) \simeq A_5.$$

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No other suitable projective incidence graphs

Vertex-face graph and projectivization seems to be the only way that produces A_5 as the automorphism group.

Projective 1-skeleton (vertex-edge) graph has $|Aut| = 720$.
2-neighbourhood of a vertex has symmetry group Σ_5. Thus there are $6 \cdot 120 = 720$ automorphisms. Aut increases after projectivization.

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A modification of the projective vertex-face graph of the icosahedron and A_4

Since A_5 has subgroups isomorphic to A_4, we can try to modify Π_I — add edges to destroy some symmetry, so that the automorphism group of the modified graph is isomorphic to A_4.

We find generators for a subgroup $H \leq Rot(I)$, such that $H \simeq A_4$. It follows that $A_4 \simeq f(H) \leq Aut(\Pi_I)$, where $f : Rot(I) \to Aut(\Pi_I)$ is the group morphism defined in the previous proof.

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Choose a 6-subset of vertices $W = \{O, A, B, C, D, E\} \subseteq V(I)$ such that $I_1[W]$ (graph induced by W in the 1-skeleton graph of I) is isomorphic to the 5-wheel, see Fig.3.

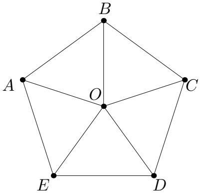

Fig.3. - $I_1[W]$.

Note that O, A, B, C, D, E in Fig.3 represent the whole $[V]$.

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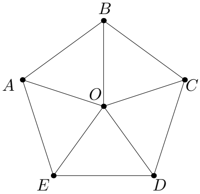

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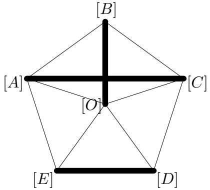
Define an undirected bipartite graph

$$\Gamma_4 = \Xi_I = ([V], [F], I_p \cup J)$$

by adding three edges to Π_I: $J = \{[A] \sim [C], [B] \sim [O], [D] \sim [E]\}$, see Fig.4, Fig.5.



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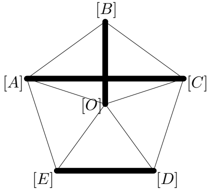

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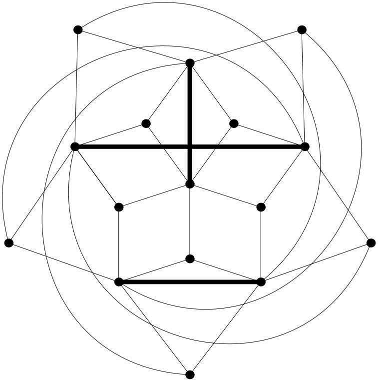

Fig.5. - Ξ_I.

A modification of the projective vertex-face graph of the icosahedron and A_4

Theorem

$Aut(\Xi_I) \simeq A_4$.

Since $E(\Xi_I) \supset E(\Pi_I)$, it follows that $Aut(\Xi_I) \leq Aut(\Pi_I)$.

Consider the subgroup $H = \langle a, b \rangle \leq Rot(I)$ generated by two rotations:

1. a - a rotation of order 2 around the line passing through the center of the edge OB and the center of I,
2. b - a rotation of order 3 around the line passing through the center of the face OCD and the center of I.

We prove that

1. $H \simeq A_4$ and
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To prove that $H \simeq A_4$ we investigate subgroups of A_5 generated by two elements of order 2 and 3.

If $H' = \langle a', b' \rangle \leq A_5$, $ord(a') = 2$, $ord(b') = 3$, then there are 3 possibilities for the isomorphism type of the functional graph ("cycle type") of the pair (a', b'):

1. $(a_1, b_1) \simeq ((12)(34), (345))$, $\langle a_1, b_1 \rangle \simeq \Sigma_3$, $ord(a_1 b_1) = 2$,
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We have that $Rot(I) = \langle a, b, c \rangle = \langle H, c \rangle$ where c is any rotation of order 5.

Since $|Rot(I) : H| = 5$ it follows that multiplication by c acts cyclically on cosets $mod\ H$ and any element of $Rot(I)$ is in form $c^n h$ where $h \in \langle a, b \rangle = H$.

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Let c be the rotation around the line passing through the center of I and O corresponding to the vertex permutation $(ABCDE)$.

$f(c)$ permutes vertex classes: $f(c)|_V = ([A][B][C][D][E])$.

The edge $[O] \sim [B]$ is the only extra edge having $[O]$ as a vertex, all edges from $[O]$ are rotationally permuted by $f(c^n)$, see Fig.4.

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$f(c)$ permutes vertex classes: $f(c)|_V = ([A][B][C][D][E])$.

The edge $[O] \sim [B]$ is the only extra edge having $[O]$ as a vertex, all edges from $[O]$ are rotationally permuted by $f(c^n)$, see Fig.4.

It follows that nontrivial elements $f(c^n)$ do not permute the three extra edges in Ξ_I.

All is proved.

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Conclusion

Number of vertices of undirected graphs having automorphism groups A_5, A_4 can be reduced to 16 by projectivization.

It can be checked by exaustive search that $\mu(A_5) = \mu(A_4) = 16$. It would be better to find a proof (not done yet).

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References I

Conder, M.,
Complete list of all connected edge-transitive bipartite graphs on up to 63 vertices, retrieved February 13, 2019, from https://www.math.auckland.ac.nz/ conder/ AllSmallETBgraphs-upto63-full.txt.

Daugulis, P.
A note on another construction of graphs with $4n + 6$ vertices and cyclic automorphism group of order 4n,
Archivum Mathematicum, vol. 53, no. 1, 2017, pp. 13–18.

References II

Daugulis, P.
16-vertex graphs with automorphism groups A_4 and A_5 from the icosahedron.
Electronic Journal of Graph Theory and Applications, Vol. 8, No. 2 (2020), pp. 211–216.

M. E. Watkins,
Graphical regular representations of alternating, symmetric, and miscellaneous small groups,
Aequationes mathematicae, **Volume 11, Issue 1** (1974), pp 40-50.