Proof is number - proposals for a research program

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Outline

Abstract

Main MSC 03B35 = General logic, mechanization of proofs and logical operations.
Additional MCS 00A30 = General, Philosophy of mathematics.

The need to develop a theory for measuring value and complexity of mathematical implications and proofs is discussed including motivations, and possible benefits. Examples of mathematical considerations are given for such a theory. Arguments supporting applications in mathematical research guidance are given. No definitions and theorems.

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History

Recall the hypothetical saying attributed to Pythagoras - **"all is number"** $\sim$ all physical objects, systems and processes may be precisely mathematically modelled. Number is a metaphor, meaning a mathematical model.

In philosophical terms mathematics — a uniform framework for performing justification/regress steps for knowledge from various areas.

Definition (philosophy)- Justification/regress step

[Justification/regress step — map (explain) information A to simpler, more fundamental information α.]

Category theory

- an intra-mathematical regress step - mapping various mathematical theories to graphs with additional structures.

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The application areas having mathematical models and being served by applied mathematics are constantly enlarging and models are getting more precise and rigorous.

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One can ask what should be the next steps of the intra-mathematical Pythagorean process.

We must look at those identifiable mathematical activities which have not so far been coordinatized and measured.

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- Mathematical expression of value of mathematical statements (theorems etc.).
- Mathematical structure of mathematical theories.

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Given a mathematical theory A (a structure describing objects of study, first or higher-order logic statements) find a mathematical object $\tau(A)$ which would be a good model of A: elements of A such as logical implications and proofs in A would be defined as substructures or quotient structures of $\tau(A)$.

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The proposed idea seems to go beyond standard algebraic logic and proof theory which deals with constructions of systems of axioms, correct statements, syntactic and language problems, expressive power problems of axiom and inference systems.

Our idea can be very vaguely compared to introducing Cartesian coordinates in the space of statements - assigning implications directions and lengths.

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This research proposal appears to be related to the lesser-known unpublished 24th Hilbert's problem - find the simplest proof of a given statement, compare different proofs, design criterions for simplicity and rigor etc., see [9].

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It would also be used to guide researchers, show them the most important research directions, problems and milestones in a rigorous and quantitative way.

Problems and proofs which are considered important, must have mathematically well defined extremal properties.

Progress of mathematics and the goal of mathematics itself (locally and globally) have to be defined as mathematical objects.

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Computerization

Computers or their future descendants will be eventually used to perform mathematical research.

Therefore we may need to create theories which would model human mathematical thinking using mathematical objects which can be processed by computers, reduce mathematical goal setting and theorem proving to computation.

If this approach is successful we may ask fundamental questions.

- What can be considered an advanced/computerized form of mathematical or general implication/consequence making? If there is such a form how it can be implemented?

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Directions and examples of mathematical considerations - Understanding implications

Proofs of mathematical statements are sequences or, more generally, networks of logical implications.

Therefore one approach to the study of proofs would be to study relatively simple logical implications and their networks.

Research may be needed to determine

- right definitions of irreducible implications,
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Inclusion of predicate supports

Traditionally, logical implications are defined as instances of a relation on logical predicates in first-order or higher-order logic using the material condition (if-then) connective $\rightarrow$.

Given two predicates $P(x)$ and $Q(x)$ defined for all $x \in X$ we say that P implies Q ($P \rightarrow Q$) if

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The support (characteristic set) $supp(A)$ of a predicate A may be defined as the set of A argument values x for which $A(x) = true$.

Implication as inclusion

Validity of a predicate implication $P \to Q$ is equivalent to set-theoretic inclusion of $supp(P)$ into $supp(Q)$: $P \to Q$ is a true statement if and only if $supp(P) \subseteq supp(Q)$.

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We could describe the implication $P \to Q$ by set-theoretical, combinatorial, algebro-geometrical, geometrical, topological and complexity-theoretical properties of the predicate support (characteristic) sets $supp(P)$ and $supp(Q)$.

Such as

- absolute and relative sizes and shapes of $supp(P)$, $supp(Q)$ and $supp(Q) \backslash supp(P)$,
- properties of the boundaries of $supp(P)$ and $supp(Q)$.

For instance, we can define that that

- the implication $P \to Q$ can be considered easy if $supp(P)$ is a relatively small, e.g. low-dimensional, subset of $supp(Q)$ (?);
- implications $P \to Q_1$ and $P \to Q_2$ can be considered distinct if $(supp(Q_1) \cap supp(Q_2)) \backslash supp(P)$ is relatively small. (?)

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Passing from semantic-specific implication making to constructing sequences of embedded sets should be considered as a computational substitution of implication making.

Coordinatization and measurement of logical implications may be related or even reduced to computational complexity if computations are involved determining the inclusion $supp(P) \subseteq supp(Q)$.

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We give a candidate definition for irreducible implications in the case of propositional logic (predicates depending on binary vectors).

Suppose $p(X_1, ..., X_n)$ and $q(X_1, ..., X_n)$ - formulas in propositional Boolean variables $X_1, ..., X_n$ and the implication $p \to q$ is true.

We call an implication $p \to q$ irreducible if the full disjunctive normal form (DNF) of q has exactly one more disjunctive term than the full DNF of p. Such an implication $p \to q$ is not a composition of two noninvertible implications.

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Complexity of formulas in propositional Boolean variables can be measured in terms of their minimal disjunctive or conjunctive forms, structure of prime implicants, Blake canonical forms, logical depth and other circuit complexity measures, see [4].

Complexity of an implication $p(X_1, ..., X_n) \to q(X_1, ..., X_n)$ can be measured in terms of changes of normal forms (disjunctive, conjunctive, minimal, Blake etc.) of p and q.

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Graph (binary relation) theory

Graphs (binary relations) are used in mathematical logic, see [8] for a recent work. Propositional fomulas have been modeled as graphs — *cographs*.

Recently there has been an attempt to encode mathematical logic "without syntax" - to define and study **combinatorial proofs** in propositional logic as graph homomorphisms of certain kind, see [7].

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Graph theory - Process graphs

AoA (Activity-on-Arc) proof graphs

AoA type proof graph $\Pi = (\Sigma, \Lambda)$ - elements of Σ are statements (which are not interpreted as implications) and directed edges in the set Λ are logical implications.

Assume that any edge of a proof graph Π is given a weight which measures the complexity or some other well defined property of the corresponding implication.

Assume that we are given a directed path between two vertices P and Q having edges $e_1, e_2, ..., e_n$ with weights $w_1, w_2, ..., w_n$ which corresponds to a proof $P \to Q$. Complexity or other measure of the proof could be defined as an appropriate function of weights $w_1, w_2, ..., w_n$, for example, $\sum_i w_i$.

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Running example — Propositional logic

In the case of propositional logic, an example of edge weight corresponding to an implication $p \to q$ could be the number of set-theoretic operations necessary to produce $supp(q)$ starting with $supp(p)$.

Graph theory — Neighbourhoods

We could investigate problems such as, for example, the problem of finding all statements in a neighbourhood of the premise/axioms — within a fixed distance from a given statement or axiom.

Graph theory - Vertices with special/extremal properties as valuable statements

In general, proof modeling and, in particular, proof graph models should enable us to rigorously identify extremal statements and extremal implication steps which are relatively more or less important than others.

In particular, vertices of proof graphs having extremal properties related to connectivity, metric, centrality or other invariants may be considered as valuable "theorems".

The same arguments should identify statements (peripheral, low degree etc.) which can be considered of low value.

In our running example — low value statements would correspond to subsets in $\mathbb{F}_2^n$ that can be obtained using a small number of set-theoretic operations.

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Having fixed vertices P and Q we can study all (P, Q)-paths, e.g. we can pose the problem of finding all (P, Q)-proofs in a right sense.

We can also try to find vertices with special properties, e.g. vertices which are in more than one (P, Q)-path.

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The composition of implications can be interpreted as a binary associative operation on the set of implications.

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A mathematical theory (Σ, Λ) can also be endowed a topological space structure as follows.

Note that the implication binary relation $\rightarrow$ is a preorder relation - it is obviously reflexive and transitive.

We can view the implication relation as a specialization preorder for the Alexandrov topology τ on Σ corresponding to $\leftarrow$: the open sets for τ are the upper sets with respect to the relation $\leftarrow$, see [3].

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Proof bundles

If we have two predicates $P(x)$, $Q(x)$ where $x \in X$ and an implication or proof $f : P \to Q$ which is true for every $x \in X$ then complexity of proofs and proofs themselves may be different for different $x \in X$.

Such situations may be considered using topological analogy with topological bundles, the set X being the base and the proof f_x for each $x \in X$ being the fiber.

This may generalize the standard "proof-by-case" method.

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Mathematical results and theories should be analyzed with respect to existence of Noetherian induction proofs.

Suppose the statement $\forall\ x \in X\ \ P(x)$ is true. Does there exist a well-founded relation $R \subseteq X \times X$ (well-founded = no infinite descending chains) such that the statement can be proved using Noetherian (structural) induction on R ?

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Some concrete proposals

We can formulate a few specific initial research proposals:

- analyze the body of facts of some classical domain (for example, Euclidean geometry) with respect to the implication modelling and Hilbertian simplicity idea, create a database of all nonequivalent logical steps or deduction rules,
- analyze the body of facts of some classical domain with respect to structural induction, create a database of all nonequivalent induction arguments,
- classify invariants and object properties in a mathematical domain such as, for example, graph theory, with respect to computational complexity (e.g. polynomial or NP-complete) of decision problems, study the network of polynomial reductions,
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Conclusion

We have given a number of arguments which describe a research proposal in mathematical logic.

It can be defined as faithful mathematical representation of implications and proofs.

The main argument is a desire to formalize, map to simpler mathematical objects and measure logical implications, to make nontrivial and creative mathematical theorem proving a computation.

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