Links between Latin squares, nets, graphs and groups:

Work inspired by a paper of A. Barlotti and K. Strambach

A Talk by M. Klin on September 2025
based on work by:

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CONTENT

1. Preamble

<u>Formal main goal:</u>

To present an infinite series of PROPER LOOPS which like groups have a ”high symmetry“.

<u>Additional goal:</u>

Using this opportunity to consider with renewed rigour various links of Latin squares with other combinatorial, geometrical and algebraic structures.

Various points of view:

- Latin square as an array
- Latin square as Cayley table of a quasigroup
- Latin square as $3 \times n$ orthogonal array
- Latin square as ternary relation, that is a collection of triples
- 3-net
- Transversal design
- Amorphic association scheme
- Strongly regular graph

and so on

A methodological principle

Assume we investigate a structure γ with rich symmetry.
Then:

- Describe $G = Aut(\gamma)$
- Get a "natural" action for G
- Induce from this natural action of G all other actions related to γ
- Get in these new terms a nice interpretation of γ

Example 1:

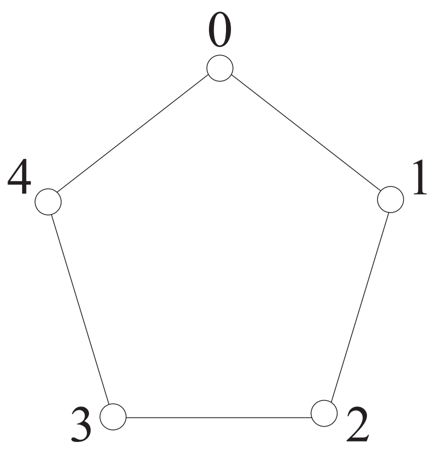
Block design $D = (V, B)$ with the parameters

$$v = 6, b = 10, k = 3, r = 5, \lambda = 2$$

(the smallest non-trivial BIBD).

It turns out that $Aut(D) \cong A_5$ is a group of order 60.

Consider the auxiliary structure pentagon C_5:



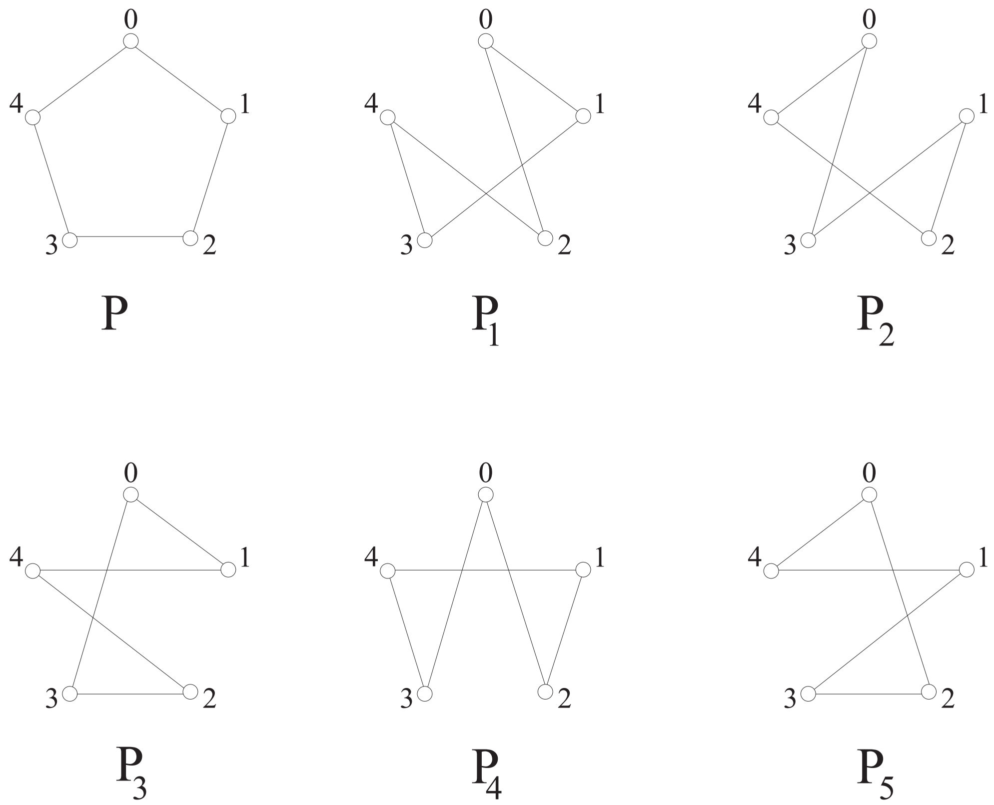
$$Aut(C_5) \cong D_5,\ |D_5| = 10$$

$$D_5 = < (0, 1, 2, 3, 4), (1, 4)(2, 3) >$$

The group D_5 consists of even permutations only.

The orbit $C_5^{A_5}$ has length $\frac{60}{10} = 6$.

Here it is:



$\mathcal{P} = \{P_0, P_1, P_2, P_3, P_4, P_5\}$

$\mathcal{B} = \{b_0, b_1, \dots, b_9\}$ - edge set of K_5 with

$$
\begin{aligned}
&b_0 = \{0,1\},\ b_1 = \{0,2\},\ b_2 = \{0,3\},\ b_3 = \{0,4\},\\
&b_4 = \{1,2\},\ b_5 = \{1,3\},\ b_6 = \{1,4\},\ b_7 = \{2,3\},\\
&b_8 = \{2,4\},\ b_9 = \{3,4\}.
\end{aligned}
$$

Incidence: edge is included in a pentagon

Final list of D:

$$
\begin{aligned}
b_0 &= \{P_0, P_1, P_3\} \\
b_1 &= \{P_1, P_4, P_5\} \\
b_2 &= \{P_2, P_3, P_4\} \\
b_3 &= \{P_0, P_2, P_5\} \\
b_4 &= \{P_0, P_2, P_4\} \\
b_5 &= \{P_1, P_2, P_5\} \\
b_6 &= \{P_3, P_4, P_5\} \\
b_7 &= \{P_0, P_3, P_5\} \\
b_8 &= \{P_1, P_2, P_3\} \\
b_9 &= \{P_0, P_1, P_4\}
\end{aligned}
$$

All properties are quite visible from this interpretation!

We will come back to this example.

2. Main definitions

<u>Latin square:</u> (naive way)

An $n \times n$ array L with n different entries, $n \geq 2$, such that each entry occurs exactly once in any row and any column.

As a rule we set

$$R = C = S = [1, n] = \{x \in \mathbb{N} | 1 \leq y \leq n\}$$

(R, C, S means Rows, Columns, Symbols)

<u>Reduced</u> Latin square L:

in the first row and the first column the elements $1, 2, \ldots, n$ occur in the natural order.

A quasigroup $(Q, \cdot)$ is a set Q with a binary operation "$\cdot$" such that for all $a, b \in Q$ the equations

$$a \cdot x = b \text{ and } y \cdot a = b$$

have a unique solution in Q.

Latin square = Cayley table of a quasigroup

A loop is a quasigroup with an identity element $e \in L$. Usually e is identified with 1.

Then:

$$\text{Loop} = \text{reduced Latin square}$$

A group is an associative loop.

Latin square as a <u>ternary relation</u> $L \subseteq [1,n]^3$, such that $|L| = n^2$ and the sets

$$
\begin{aligned}
L_{1,2} &= \{(i,j) | (i,j,k) \in L\}, \\
L_{1,3} &= \{(i,k) | (i,j,k) \in L\}, \\
L_{2,3} &= \{(j,k) | (i,j,k) \in L\},
\end{aligned}
$$

have n^2 distinct elements.

<u>Particular case of a group H</u>

$$\{(i,j,k) | i,j,k \in H, ijk = 1\},$$

where 1 is the identity element of H.

An orthogonal array $OA(n,3)$ of order n and depth 3 is a $3 \times n^2$ array with entries from $[1,n]$, such that for any two rows of the array the n^2 vertical pairs occuring in this array are different.

Remark:

Ternary relation is a set

Orthogonal array is a "vertical ordered" representation of the set

A <u>3-net of order n</u> is an incidence structure $\gamma = (\mathcal{P}, \mathcal{L})$, which consists of an n^2-element set $\mathcal{P}$ of points and a $3n$-element set $\mathcal{L}$ of lines.

There is a partition $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 = \mathcal{L}$ into disjoint families of parallel lines (directions).

Axioms:

1) every point is incident with exactly one line of each family $\mathcal{L}_i, i = 1, 2, 3$.

(2) two lines in different families intersect in exactly one point

(3) two different lines in the same family do not intersect

(4) there exist three lines in three distinct families which are not incident with the same point

Latin square $L \to$ 3-net $\gamma(L)$:

Points are cells of L

Directions correspond to horizontal lines, vertical lines, and the lines occupied by the same element

A 3-net is a uniform and regular incidence structure with

$$v = n^2, b = 3n, k = n, r = 3.$$

This is a particular case of a partial geometry.

We can get (many) Latin squares from a given 3-net.

Consider the dual structure $\gamma^T = (\mathcal{L}, \mathcal{P})$ which has $\mathcal{L}$ as points, $\mathcal{P}$ as lines and the transposed incidence relation.

The three families of points are called <u>groups</u> (unfortunate term), each of cardinality n, and there are n^2 blocks (lines):

$$v = 3n, b = n^2, k = 3, r = n.$$

The obtained structure is called a <u>transversal design</u> $TD(3, n)$.
(which is by our definition a resolvable design)

One more axiomatization in terms of association schemes:

$\mathcal{M} = \mathcal{M}(L) = (\Omega, \{R_0, R_1, R_2, R_3, R_4\})$, where $|\Omega| = n^2$ and a family $\{R_i | 0 \leq i \leq\}$ is a partition of Ω^2, which satisfies certain axioms.

A strongly regular graph $\Gamma = LSG(L)$ with parameters

$$v = n^2, k = 3(n-1), \lambda = n, \mu = 6.$$

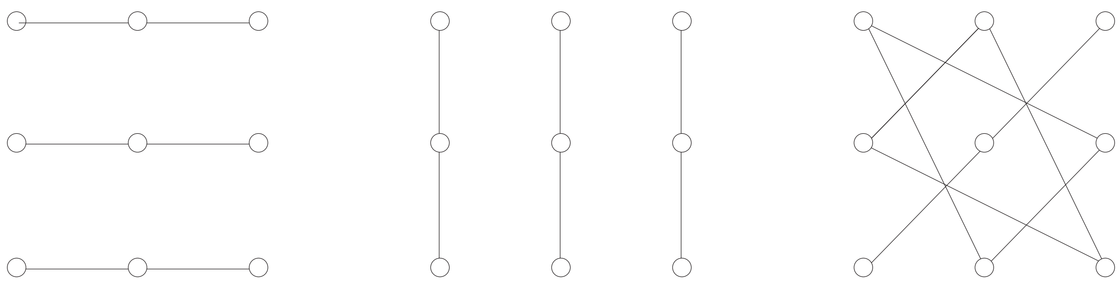
The vertices are the cells of L.

Adjacency: two cells are in the same row, in the same column or are occupied by the same symbol.

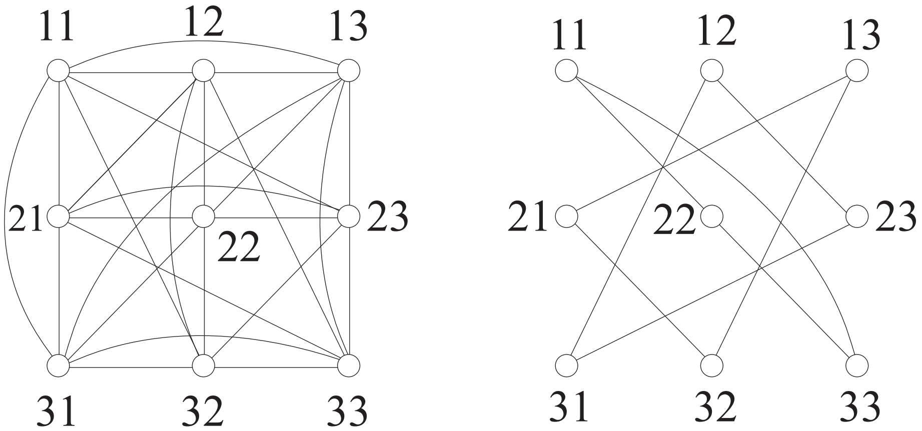
Example 2:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}$$
$$\mathbb{Z}_3$$

3-net $\mathcal{N}(\mathbb{Z}_3)$:



Strongly regular graph $\Gamma = LSG(\mathbb{Z}_3)$:



Γ $\qquad$ $\overline{\Gamma} = 3 \circ K_3$

$$v = 9, k = 6, \lambda = 3, \mu = 6$$

Classification

- reduced squares
- isotopy classes
- types
- quasigroups
- loops
- main classes

Two Latin squares L_1 and L_2 are in the same main class if and only if the corresponding 3-nets are isomorphic, i.e. $\mathcal{N}(L_1) \cong \mathcal{N}(L_2)$.

Classics and Folklore

<u>Lemma 1:</u>
Let L be a Latin square and $\Gamma = LSG(L)$. If $n \geq 5$ then the cliques of Γ necessarily correspond to the lines of an associated 3-net $\mathcal{N}(L)$.

<u>Lemma 2:</u>
For $n \geq 5$ we can reconstruct the 3-net $\mathcal{N}(L)$ uniquely from the graph $\Gamma = LSG(L)$.

<u>Lemma 3:</u>
For $n \geq 5$ we have

$$Aut(LSG(L)) = Aut(\mathcal{N}(L)) = \Sigma(L).$$

The group $\Sigma(L)$ is sometimes called the *collineation group* of L.

Lemma 4:

Take a Latin square L as a Cayley table of a group H. Then

$$Aut(\mathcal{N}(L)) \cong (H^2 : Aut(H)).S_3$$

Theorem 5:

Let L be a group Latin square corresponding to a group H. Assume $|H| \geq 5$. Then

$$Aut(LSG(L)) \cong (H^2 : Aut(H)).S_3$$

Corollary 6:

Let H be a group of order n, $\Gamma = LSG(H)$. Then

a) $Aut(\Gamma)$ is a transitive permutation group of degree n^2,

b) $Aut(\Gamma)$ contains a regular sungroup H^2 of order (and degree) n^2.

Proposition 7:

Let H be a group of order n and let Q be a loop of order n. Then $H \cong Q$ (isomorphic as loops) if and only if the corresponding 3-nets $\mathcal{N}(H)$ and $\mathcal{N}(Q)$ are isomorphic.

Corollary 8:

a) If H_1 and H_2 are nonisomorphic groups of order n, then $LSG(H_1) \not\cong LSG(H_2)$.

b) If a Latin square L does not appear in a main class of any group, then $LSG(L)$ is not isomorphic to any Latin square graph over a group.

4. More examples

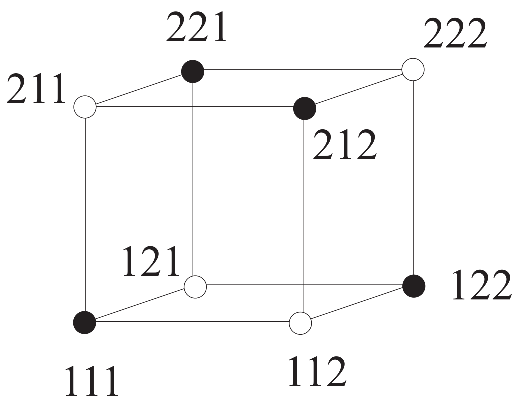
Example 3: The case $n = 2$

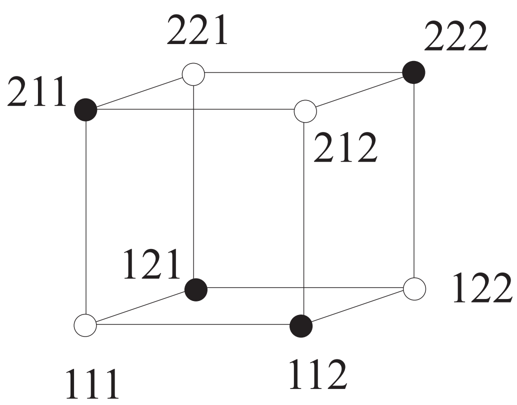
There are two Latin squares:

1	2
2	1

2	1
1	2

They can be regarded as cocliques in the 3-dimensional cube $Q_3 = H(3, 2)$.

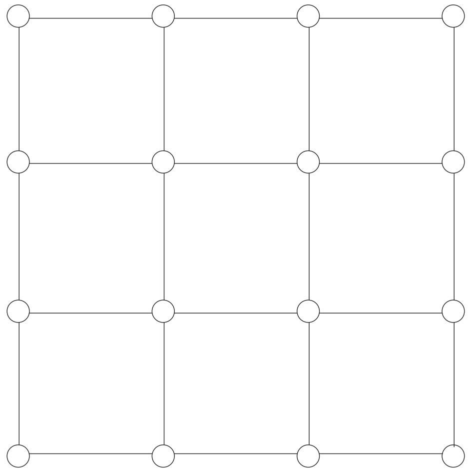




(Similar situation with Hamming graph $H(3, n)$ for arbitrary n.)

Example 4: The case $n = 4$

a) $\Gamma_1 = LSG(E_4) \cong L_2(4)$



$L_2(4)$

$Aut(L_2(4)) \cong S_2 \wr S_4$ (wreath product) is a group of order $2 \cdot (4!)^2 = 1152$.

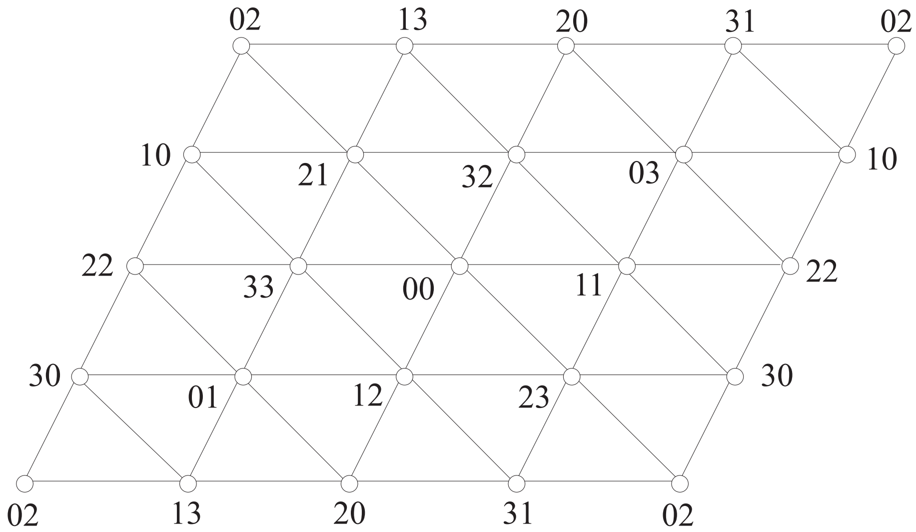
According to Lemma 4, $Aut(\mathcal{N}(E_4))$ is a group of order $4^2 \cdot 3! \cdot 3! = 576$. It is a proper subgroup of the above group.

There are two ways to reconstruct a net from a graph (the assumption $n \geq 5$ is important!).

b) $\Gamma_2 = LSG(\mathbb{Z}_4) \cong \overline{Sh}$

Sh is the famous Shrikhande graph.

$Aut(\mathcal{N}(\mathbb{Z}_4)) = 4^2 \cdot 2 \cdot 3! = 192.$



The stabilizer of a vertex is D_6 of order 12. Thus, $Aut(Sh)$ is a group of order $16 \cdot 12 = 192$. Here Theorem 5 is also valid.

The graphs Γ_1, Γ_2 describe the only two main classes for $n = 4$.

Example 5: The case $n = 5$

Still there are two main classes

$\Gamma_1 = LSG(\mathbb{Z}_5)$ is isomorphic to the Paley graph over the field F_{25},
$\Gamma_2 = LSG(Q)$ for a certain proper loop Q of order 5.

This is the smallest case when we have a proper loop, and when $Aut(\Gamma)$ is an intransitive permutation group.

$Aut(\Gamma_1)$ is a group of order $5^2 \cdot 4 \cdot 3! = 600$.

$Aut(\Gamma_2) \cong (S_4 + S_3)^{\mathrm{pos}}$ is of order $\frac{1}{2} \cdot 4! \cdot 3! = 72$.
(A nice model is available.)

Example 6: The case $n = 6$

There are 12 main classes. Three of them have a transitive automorphism group:

$AutLSG(\mathbb{Z}_6)$ is a group of order $6^2 \cdot 2 \cdot 3! = 432$,

$AutLSG(S_3)$ is a group of order $6^2 \cdot 3! \cdot 3! = 1296$,

$AutLSG(Q_6)$ has order 648 (second part of the talk).

All other groups are intransitive.

A nice model for a main class with the group $S_5 \times S_2$ of order 240 (in terms of Example 1) is available.

n	main classes	types	isotopy classes	loops	reduced squares	quasigroups	total amount of squares
1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	2
3	1	1	1	1	1	5	12
4	2	2	2	2	4	35	576
5	2	2	2	6	56	1411	161280
6	12	17	22	109	9408	1130531	812851200

Data for $n \geq 6$

5. The remark of Barlotti and Strambach

We can now express our interest in "group-like quasigroups" in a more concrete manner as it was done by

A. Barlotti & K. Strambach
(Adv. in Math. 49 (1983), 1–105).

They wrote on page 79 of their survey paper:

> *We were not able to decide whether there exists a proper finite loop having a sharply transitive group of collineations.*

(compare Corollary 6 b)

As far as we know such an example did not appear in evident form in the literature.

The answer is surprisingly simple.

<u>Proposition 9:</u> (HK)

Consider the following Latin square Q_6:
(No. 3.1.1 in Dénes & Keedwell, 1974)

1	2	3	4	5	6
2	3	1	5	6	2
3	1	2	6	4	5
4	6	5	2	1	3
5	6	4	3	2	1
6	5	4	1	3	2

Then:

(a) The main class of Q_6 does not contain a group;

(b) $G = Aut(\Gamma(Q_6))$ is a transitive permutation group of degree 36 and order 648;

(c) G has a regular subgroup.

Original proof: brute force computations with the aid of GAP, GRAPE and nauty.

6. Computer aided answer

... was obtained as a byproduct of a general problem considered by Aiso Heinze in his Ph.D Thesis (2001):

The complete classification of all partial difference sets over "small"groups (of order ≤ 49).

Recall that a partial difference set means a connection set of a strongly regular Cayley graph.

(Catalogues of Ted Spence were used for the general case, however, they are not requested in order to get this particular result.)

The use of GAP allowes us to obtain a set of generators of our group G of order 648 as well as a description of two (up to isomorphism) regular subgroups H_1 and H_2 of order 36.

It turns out that $H_2 \cong S_3 \times S_3$.
Now we guess what the structure of G is :

Is it true that $G \cong (S_3 \wr S_3)^{\mathrm{pos}}$ (i.e., a subgroup of even permutations in the wreath product of S_3 with S_3)?

$\rightarrow$ Yes, answers GAP.

Is it true that $G_x \cong D_9$ (here D_9 is the dihedral group of degree 9 and order 18)?

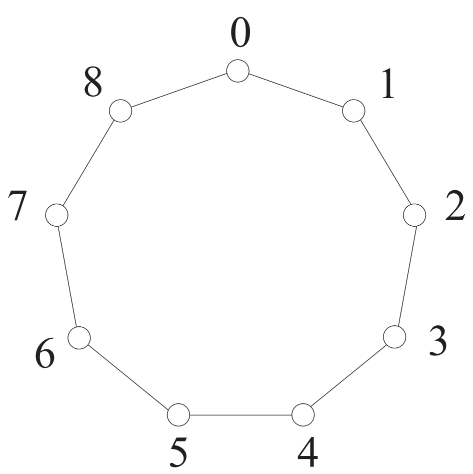
$\rightarrow$ Yes, answers GAP.

Now we are prepared to use another computer package COCO (I. A. Faradžev, K., Moscow, 1990–1992)

<u>Input:</u>

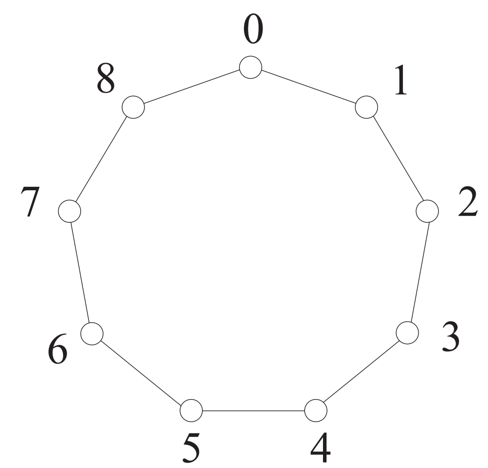
a) a set of generators for G:
$g_1 = (0, 1, 2, 3, 4, 5, 6, 7, 8)$,
$g_2 = (0, 3, 6)$,
$g_3 = (0, 3)(1, 4)$,
$g_4 = (0, 1)(3, 4, 6, 7)$.

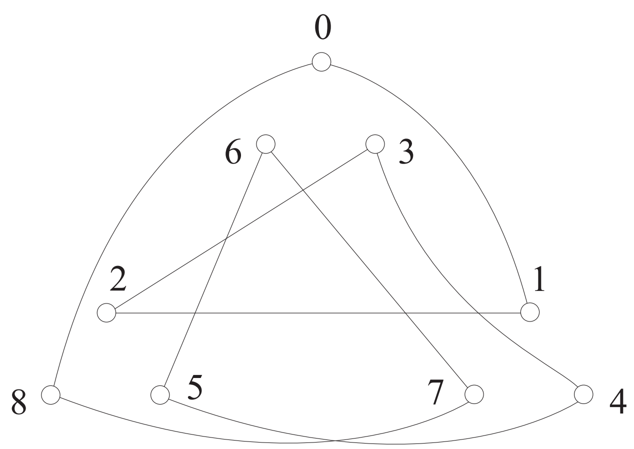
b) the 9-gon C_9, such that $Aut(C_9)$ is D_9 and $Aut(C_9) \leq G$.

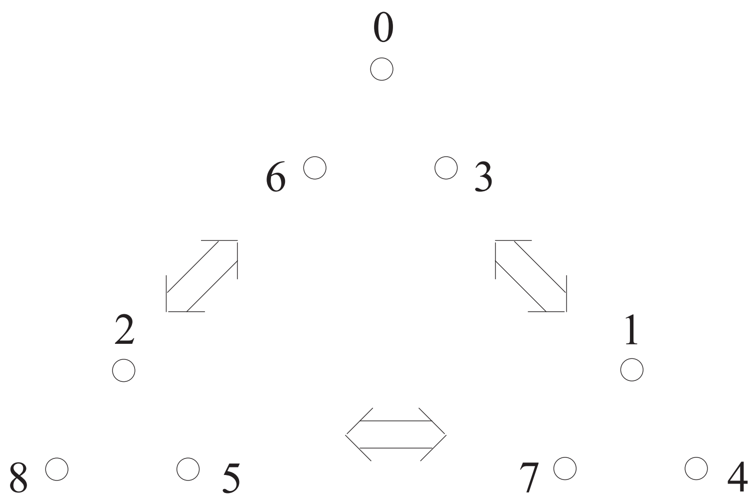


C_9

Why this set of generators for G?







(0,1,2,3,4,5,6,7,8),
(0,3,6),
(0,3)(1,4),
(0,1)(3,4,6,7).

Output:

- transitive action of G on all different copies of D_9 with respect to permutations from G; this is an action of degree $\frac{648}{18} = 36$;
- 2-orbits of this action (subdegrees are 1,1,1,6,9,9,9);
- intersection numbers of a corresponding association scheme with 6 classes;
- all mergings of classes of this association scheme.

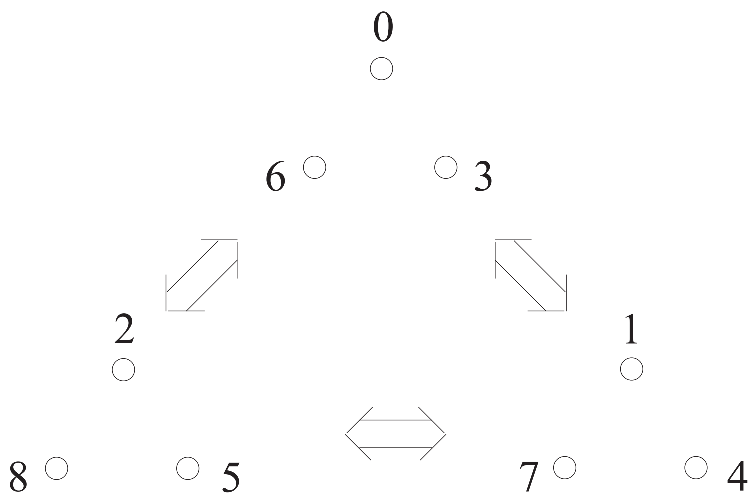
In particular, we get three isomorphic copies of the graph

$$\Gamma = LSG(Q_6)$$

which is a strongly regular graph with the parameters

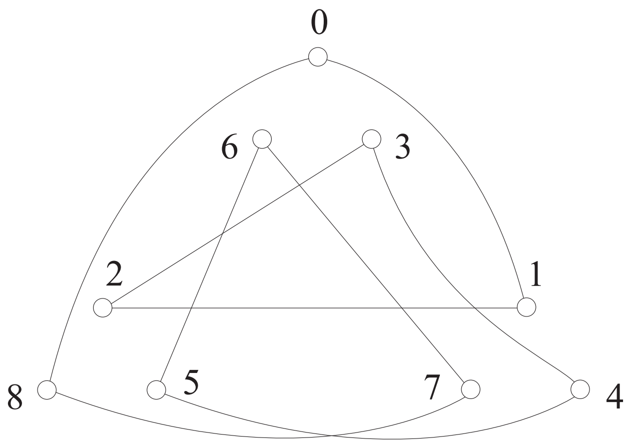
$$v = 36, k = 15, \lambda = 6, \mu = 6.$$

The first step to a computer free interpretation is an *explanation* of some of the results obtained by COCO.
Consider the auxiliary graph $\Delta = \overline{3 \circ K_3}$.



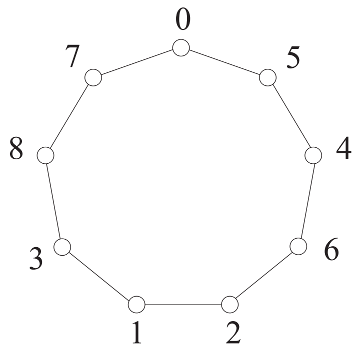
$\Leftrightarrow$ means 9 edges of $K_{3,3}$

Δ has $3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 72$ spanning subgraphs which are Hamiltonian cycles in Δ.



G has two orbits of length 36 in its action on these 72 cycles (G is a "half" of $Aut(\Delta)$, while all permutations in D_9 are even).

Solve a problem of a *famous donkey*:
Select one of these two orbits (take the one which includes our canonical C_9).
Get representatives of the neighbours for 6 classes:



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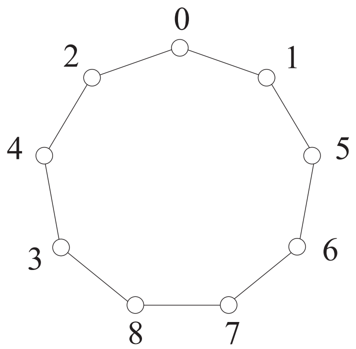

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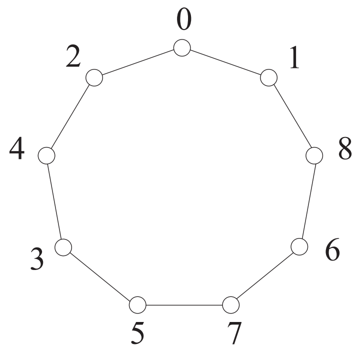

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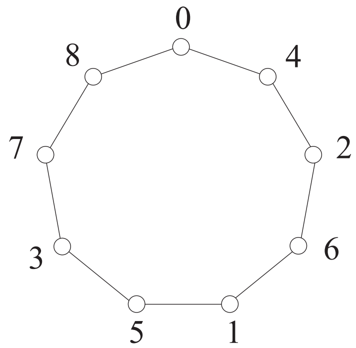

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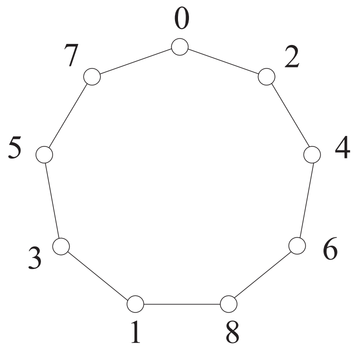

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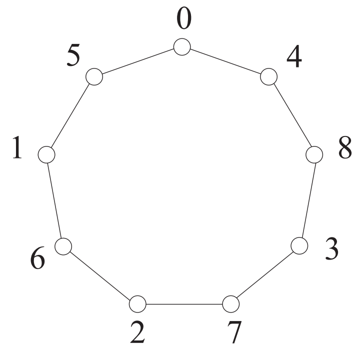


1

Now we have three possibilities to merge a relation of valency 6 with a relation of valency 9. Each time we get a desire strongly regular graph of valency 15.

The proof of the fact that we indeed are getting a desired SRG, in principle, can be managed by hand computations.

However, in principle, there is no essential difference with the strict use of a computer.

An honest interpretation is needed!

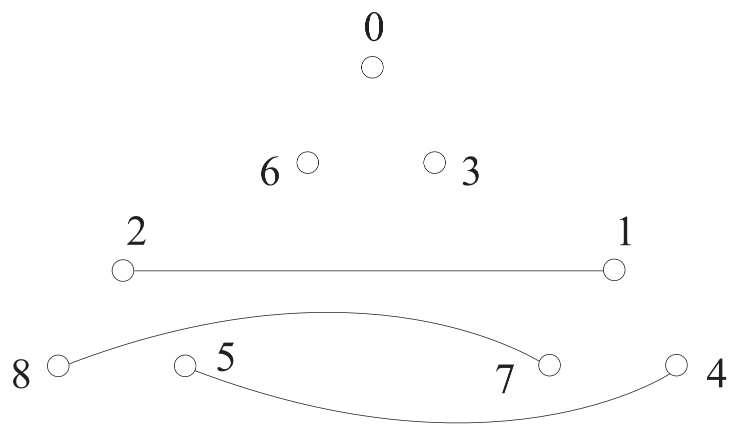
7. Computer free interpretation

General idea:

- start with a Latin square,
- get a corresponding 3-net with 36 points and 18 lines,
- this is a partial geometry,
- a dual structure is also a partial geometry, namely a transversal design $TD(3, 6)$ such that
  - there are 18 points,
  - the points are partitioned into 3 classes, each of size 6,
  - there are 36 blocks, each of size 3 (groups of the TD),
  - every unordered pair of elements from the point set is either contained in exactly one group or in exactly one block, but not both.

Let us consider once more the graph Δ.

Here is one "partial one-factor F" in Δ.



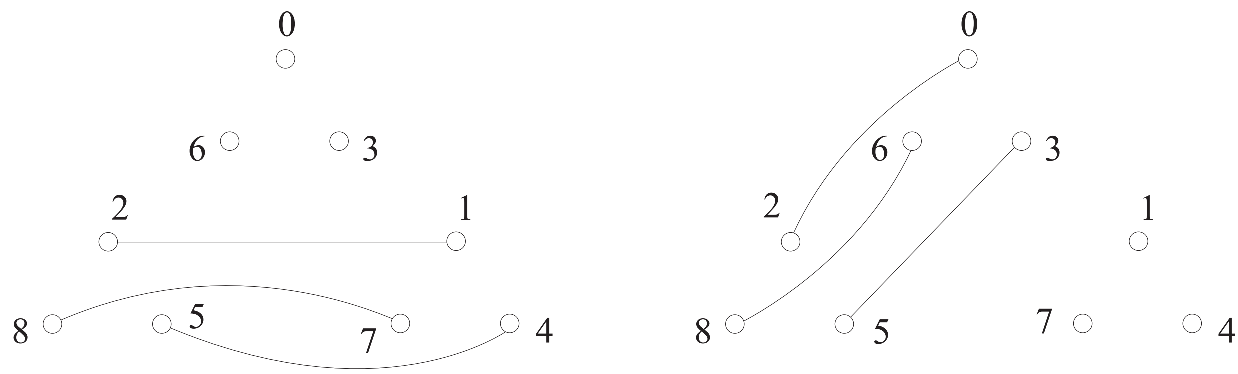
Clearly, we have $3 \cdot 3! = 18$ one-factors.

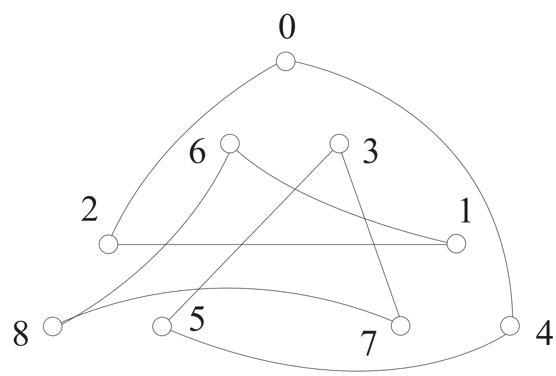
- These are points of $TD(3,6)$,
- Blocks are 36 selected cycles in Δ,
- Incidence is provided by the usual inclusion.

Most of the axioms of $TD(3,6)$ are evidently satisfied.

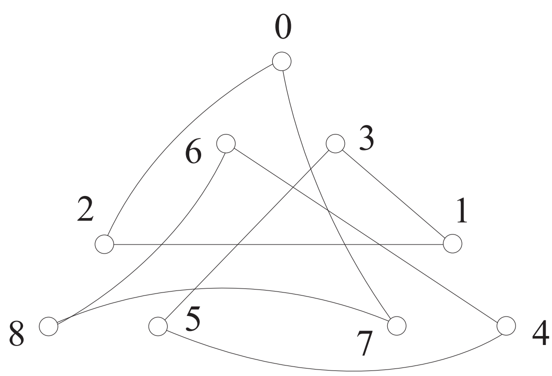
In particular, each of the three groups consist of 6 one-factors, e.g., between $\{0, 3, 6\}$ and $\{1, 4, 7\}$ (or the two other options).

It remains to check that two one-factors in distinct groups appear in exactly one cycle from a selected orbit of size 36.





C'



C''

$$\begin{array}{rlcl} C': & 0,4,5,3,7,8,6,1,2 & \quad & g : C' \to C'' \\ C'': & 0,7,8,6,4,5,3,1,2 & \quad & g = (4,7)(5,8)(3,6) \\ & & & g \text{ is odd} \end{array}$$

It is clear from the construction that

$$Aut(TD(3,6)) \geq G.$$

Some brute force inspection (using the base of a group) allows to show that we have equality.

Thus, $Aut(TD(3,6)) = Aut(LSG(Q))$ is a group of order 648.

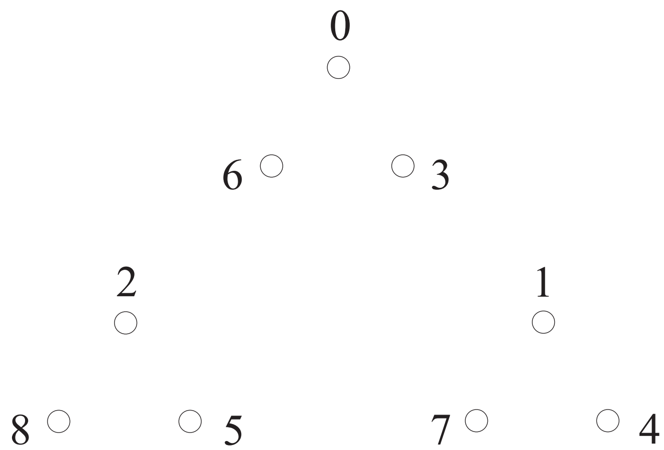
Recall that

$$|Aut(LSG(\mathbb{Z}_6))| = 6^2 \cdot 2 \cdot 6 = 432$$
$$|Aut(LSG(S_3))| = 6^2 \cdot 6 \cdot 6 = 1296.$$

Therefore, we claim that our Latin square Q (which is now hidden) is not coming from a group (i.e., does not belong to a main class of a group).

It remains to show that G has a regular subgroup in its action on 36 blocks of $TD(3,6)$.

Let us switch to the original action of G on 9 points.



Consider $H_1 \times H_2$ with

$$H_1 =< (0,3,6),(0,3)(2,5) >,$$
$$H_2 =< (1,4,7),(1,4)(2,5) > .$$

It is clear that:

a) $\lvert H\rvert = 36$ b) $H \leq G$ c) no C_9 which is preserved by H	$\Longrightarrow$	H acts regularly on 36 points

Exceptional quasigroup $\mathbf{Q_6}$ revisited

We start from $\gamma = TD(3,6)$.
- Assign labels to what will be R, C, S.
- Read the Latin square!

There is a lot of freedom.
We use special labels (elements of S_3).

<u>Main idea:</u>

1. one-factors $\leftrightarrow$ elements of S_3,
2. two one-factors define uniquely a Hamiltonian cycle,
3. an obtained Hamiltonian cycle "automatically" defines a third one-factor as a product of the first and the second one,
4. we get a loop.

Here is the result:

	e	(a,b,c)	(a,c,b)	(a,b)	(b,c)	(a,c)
e	e	(a,b,c)	(a,c,b)	(a,b)	(b,c)	(a,c)
(a,b,c)	(a,b,c)	(a,c,b)	e	(b,c)	(a,c)	(a,b)
(a,c,b)	(a,c,b)	e	(a,b,c)	(a,c)	(a,b)	(b,c)
(a,b)	(a,b)	(a,c)	(b,c)	(a,b,c)	e	(a,c,b)
(b,c)	(b,c)	(a,b)	(a,c)	(a,c,b)	(a,b,c)	e
(a,c)	(a,c)	(b,c)	(a,b)	e	(a,c,b)	(a,b,c)

Reinterpretation:

Let $s := (a, b, c)$. Then

$$x \circ y = \begin{cases} xy, & \text{if } \operatorname{sign}(x) = \operatorname{sign}(y) = 1, \\ xy, & \text{if } \operatorname{sign}(x) \cdot \operatorname{sign}(y) = -1, \\ xys, & \text{if } \operatorname{sign}(x) = \operatorname{sign}(y) = -1. \end{cases}$$

9. Infinite series

Our Latin square Q_6 and its group (implicitely) are known for a very long time (see references).

In fact, we can give a nice direct explanation of Q_6 as "twisted" dihedral group $D_3 = S_3$ of order 6.

Starting from this explanation we can describe an infinite series of examples providing a positive answer on the question of Barlotti and Strambach.

These examples are based on a similar twisted dihedral group D_p of order $2p$, where p is a prime and $p \equiv 3 \pmod 4$.

The group $Aut(LSG(Q))$ for this case appears to be $(S_3 \wr D_p)^{\mathrm{pos}}$ of order $\frac{1}{2} \cdot 3! \cdot (2p)^3 = 24p^3$.

We use similar arguments to prove that we still have a positive answer on the question by Barlotti-Strambach.

The case $p = 7$ plays a similar friendly role. We implicitely detect the existence of a quasigroup which will be denoted by Q_{2p}.

We can write it explicitely.

Consider $(D_{2p}, \cdot)$, $C_p < D_p$, $C_p =< a >$ and define $(Q_{2p}, \circ)$ as follows:

$$x \circ y = \begin{cases} xya, & \text{if } x \text{ and } y \text{ are odd,} \\ xy, & \text{otherwise} \end{cases}$$

Moreover, in a similar way a twisted quasigroup Q_{2p} may be defined for $p \equiv 1 \pmod 4$.

We have the desired examples for the cases $p = 5$ and $p = 13$.

The proof should be slightly modified.

10. Discussion

What is discovered:

(Example of Q_6 is well-known; infinite series of loops was also subject of various investigations in loop theory.)

New:

- Properties of loops
- Links between various combinatorial structures
- approach for the investigation of ”highly symmetrical“ loops.

There are also attractive new perspectives for further investigations.

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References (first round)

- G. Tarry, 1899–1900
- E. Schönhardt, 1930
- S. M. Jacob, 1930 + A. Sade, 1948
- R. A. Fisher & F. Yates, 1930
- A. A. Albert, 1944
- P. N. Saxena, 1950
- R. Artzy, 1963
- B. F. Bayant & H. Schneider, 1966
- A. Sade, 1971
- D. McCarthy, 1976
- J. Dénes & A. D. Keedwell, 1974
- D. Betten, 1984
- B. McKay & W. Myrvold, 2003
- J. J. Gol'fand, K. & I. A. Faradžev, 1985

References (second round)

- A. Suschkewitsch, 1929
- R. Artzy, 1960
- J. Singer, 1960
- E. L. Wilson, 1966
- H. Zassenhaus, 1968
- R. J. Wilson, 1974
- E. G. Goodaire & D. A. Robinson, 1982
- D. J. Finney, 1982
- L. J. Brant & G. L. Muller, 1986
- Ch. Praeger, 1989
- H. D. Pflugfelder, 1990
- R. Bailey, 1991
- K. Kunen, 2000
- J. Hall & G. P. Nagy, 2001
- A. Sprague, 1982
- E. Moorhouse, 1991
- L. Babai, 1985